A Brief History of Catalan's Conjecture

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Abstract
In 2014 one celebrates 200 years since Eugene Catalan's birth. In 1844, he stated the result that has been afterwards called "Catalan's Conjecture". In this paper, the authors present the main results that led to the proof of the conjecture.
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Since 2011 at the 200th anniversary of Evariste Galois birth, the members of the Department of Mathematics and Informatics from Spiru Haret University have started a seminary called *Moments and Personalities in the History of Mathematics*, in which one celebrates and remembers great ideas and mathematicians. In 2014 it was celebrated 200 years since Eugene Charles Catalan's birth and on April 28 2014 Rodica Ioan gave a talk on Catalan's life and activity. In this paper the authors, review the main mathematical results obtained about the conjecture, until its recent proof.

1. Crelle's Journal

In 1826, the German mathematician August Leopold Crelle (1780-1855) begins to print in Berlin the *Journal fűr die reine und angewandte Mathematik* that has been also known under the name of *Crelle's Journal*.

The journal has published some of the papers of great mathematicians, such as Euler, Abel, Cantor, and up to nowadays has remained one of the most prestigious journal of mathematics.

In 2012, its impact factor of 1.253 places the journal on the 32nd rank of the Web of Knowledge ISI list.

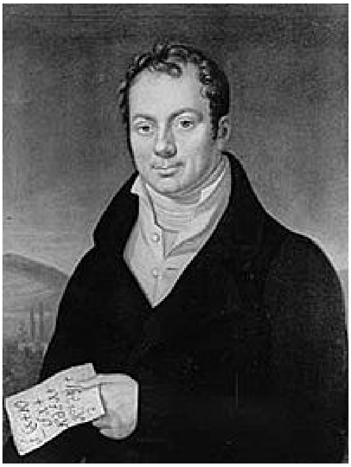

Fig. 1. August Leopold Crelle [11]

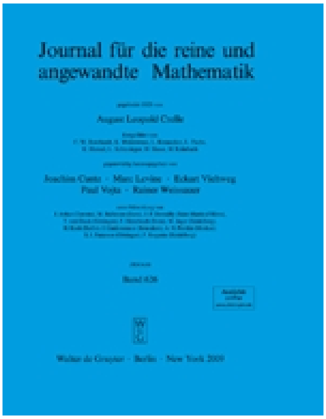

Fig. 2. Journal fűr die reine und angewandte Mathematik [12]

In 1844 Catalan [3] wrote to Crelle the following:

Je vous prie, Monsieur, de vouloir bien énoncer, dans votre recueil, le théorème suivant, que je crois vrai, bien que je n'aie pas encore réussi à le démontrer complètement: d'autres seront peut-être plus heureux:

Deux nombres entiers consécutifs, autres que 8 et 9, ne peuvent être des puissances exactes; autrement dit: l'équation

$$x^m - y^n = 1,$$

dans laquelle les inconnues sont entières et positives, n'admet qu'une seule solution.

Even if Catalan states that he cannot completely prove this fact, it is only in 1885 that Calalan [4] publishes a list of results connected to the conjecture. In the mentioned paper he states, without making any proof several particular cases of the conjecture, such as

$$(x+1)^x - x^x = 1\,.$$

1. **The Conjecture before 1844**

Even if Catalan stated the result in 1844, it seems that the problem has an older origin.

Around 1320, the mathematician Levi ben Gershon (Gersonides) (1288-1344) was interested in the study of harmonical numbers, i.e. numbers of the form $2^m \cdot 3^n$ and would show [10] that the only pairs of harmonical numbers whose difference is 1 are (1,2), (2,3), (3,4) and (8,9).

Otherwise said, Gersonides solved the equations

$$3^n - 2^m = 1 \quad (1)$$
$$2^m - 3^n = 1\,. \quad (2)$$

In 1738 Leonhard Euler (1707-1783) solved the equation

$$x^2 - y^3 = 1 \quad (3)$$

proving [5] that its only solution is $x = 3$ and $y = 2$.

1. **The conjecture after 1844**

In 1850, Victor Lebesgue (1791-1875) proved [6] that the equation

$$x^p - y^2 = 1 \quad (4)$$

has no positive integer solution if p is prime.

Along the years there were obtained several results concerning particular cases of the conjecture. It is important to mention:

- Nagel (1921): $x^3 - y^n \neq 1$;
- Selberg (1932): the equation $x^4 - y^n = 1$ has no solutions for $n > 1$;
- Ko Chao (1965): the equation $x^2 - y^n = 1$ has no solutions for $n \geq 5$.

Le Veque proved [7] in 1952 a more general result: for x and y fixed the equation

$$x^m - y^n = 1$$

has at most a solution, excepting the case $x = 3$ and $y = 2$ when the equation has two solutions: (1,1) and (2,3).

Later, in 1960 J. Cassels considers $d = g.c.d.\left(x - 1,\, \frac{x^p - 1}{x-1}\right)$; according to the value of d the problem has two cases: $d = 1$ and $d = p$ prime (similarly with Fermat's Great Theorem).

Cassels [2] proved that the first case leads to a contradiction, and the results he obtained for the second case raised again the interest for the conjecture.

There were other intents to prove it and some tried to use the new computer based technologies to obtain a proof, but without any success.

In the end, the Romanian mathematician Preda Mihăilescu [9] managed to use the theory of cyclotomic fields in order to finally prove the conjecture, that would now carry the name of **Mihăilescu's Theorem**.

The proof was given in 2002 and Catalan's Conjecture's story found its end in the same journal where it begun, *Journal fűr die reine und angewandte Mathematik* in 2004, 160 years after its statement.

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