Some Consideration About Component Minimal Complete Subgraph

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Abstract
Giving a graph G we wish to determine a subgraph H in which vertices belongs at least to a loop with length equal to 3. Firstly, we give all necessary definitions used in this paper. Secondly, we build the subgraf without isolated vertices and without tree components. Then we give some results that allow us to eliminate vertices and edges but preserve loops with length equal to 3.
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1. General definitions

Firstly I wish to present some of the graph theory definitions.

Definition 1.1 *Let*

$$V = \{x_1, x_2, \ldots, x_n\}$$

be a finite and non-empty set and

$$E = \{\{x, y\} \,| x, y \in X, x \neq y\}.$$

*The pair $G = (V, E)$ is a **graph**, elements in V are named **vertices** and element in E are named **edge**.*

Definition 1.2 *Let $G = (V, E)$ be a graph. A sequence of vertices $y_1, y_2, \ldots, y_k$ is named **walk** in G if $\{y_i, y_{i+1}\} \in E$ for any $i = 1, 2, \ldots, k-1$. If $y_1 = y_k$ the walk is named **loop**. The walk (and the loop) is specified as*

$$L = [y_1, y_2, \ldots, y_k].$$

Definition 1.3 *Let $G = (V, E)$ be a graph. If $W \subset V$ and*

$$F = \{\{x, y\} \in E | x, y \in W\}$$

then the graph $H = (W, F)$ is named ***subgraph*** *of G.*

Definition 1.4 *Let $G = (V, E)$ be a graph. If for any $x, y \in V$ there exist a walk from x to y then G is named* ***connected****. If G is not connected, G is named* ***disconnected****.*

Definition 1.5 *Let $G = (V, E)$ be a graph. A subgraph $H = (W, F)$ of G which is connected and there does not exists a walk from x to y for any $x \in W$ and $y \in V - W$ is named* ***component*** *of G.*

Propozition 1.1 *Let $G = (V, E)$ be a graph. There exist a partition of V with the sets $V_1, V_2, \ldots, V_k$ ($V_1 \cup V_2 \cup \ldots \cup V_k = V$ and $V_i \cap V_j = \oslash$ for any $i, j = 1, 2, \ldots, k$, $i \neq j$) so that subgraphs $G_i = (V_i, F_i)$ are components in G.*

Definition 1.6 *Let $G = (V, E)$ be a graph with $|V| = p \leq 3$. If for any $x, y \in V$, $\{x, y\} \in E$ then G is named* ***complete*** *and we write it as K_p.*

Definition 1.7 *Let $G = (V, E)$ be a graph. If for any $x \in V$ there exist $y, z \in V$, $y \neq z$ so that $\{\{x, y\}, \{x, z\}, \{y, z\}\} \subset E$ we say that G is a* ***minimal complete graph****.*

Definition 1.8 *Let $G = (V, E)$ be a graph. If there exist $W \subseteq V$ so that the subgraph $H = (W, F)$ is a complete graph, with $|W| = p$ and for any $x \in V - W$ the subgraph $H' = (W - \{x\} .F')$ is not a complete graph, then G is named* ***K_p-maximal complete graph****.*

Definition 1.9 *Let $G = (V, E)$ be a graph. If any component in G is a minimal complete graph then G is named* ***component minimal complete graph****.*

Definition 1.10 *Let $G = (V, E)$ be a graph and $x \in V$. If we consider the set $C = \{y \in V | \{x, y\} \in E\}$ then the number $\omega(x) = |C|$ is named the* ***degree*** *of x.*

Definition 1.11 *Let $G = (V, E)$ be a connected graph. If in G does not exist loops then G is named* ***tree****.*

Definition 1.12 *Let $G = (V, E)$ be a graph and $F \subset E$. The graph $H = (V, F)$ is named* ***partial graph*** *of G.*

Based on the definition presented above, giving a graph $G = (V, E)$ we wish to determine a subgraph H of G so that H is a component minimal complete subgraph, if such a graph exists.

2. **Preliminary conditions and eliminations**

This section is dedicated to preliminary condition and possible elimination so that component of obtained subgraph could be minimal complete graph.

Let CMC_G designate the component minimal complete subgraph for the graph G, if such a subgraph exists.

Note 2.1. *If $G = (V, \oslash)$ is a graph without edges it is obvious that G*

can not be a minimal complete graph, so our problem has no solution.

Condition 2.1. *Let $G = (V, E)$ be a graph. We consider the set*

$$X = \{x \in V | \omega(x) = 0\}.$$

If $X = V$ then the determination of component minimal complete subgraph of G has no solution.

Proposition 2.1. *Let $G = (V, E)$ be a graph and*

$$X = \{x \in V | \omega(x) = 0\}$$

with $V - X \neq \oslash$. Let consider the subgraph $H = (V - X, F)$ of G. Then, G has a component minimal complete subgraph if and only if H has a component minimal complete subgraph.

In addition, we have

$$CMC_G = CMC_H.$$

Proofs. Firstly, if G is a connected graph with $|V| > 1$, it follows that $X = \oslash$ and so $G = H$. So $CMC_G = CMC_H$ is obvious.

Secondly, we must observe that, because H is a subgraph of G, CMC_G is a subgraph in G and CMC_H is a subgraph in H, it follows that

$$CMC_H \subseteq CMC_G.$$

Let now consider that G is a disconnected graph and the component of G are $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. We have the following cases:

Case 1. $|V_1| = |V_2| = 1$.

Because $|V_1| = 1$, $V_1 = \{x_1\}$ with $\omega(x_1) = 0$ and so $x_1 \in X$.

Because $|V_2| = 1$, $V_2 = \{x_2\}$ with $\omega(x_2) = 0$ and so $x_2 \in X$.

Because $V_1 \cup V_2 = V$ it follows that $V = \{x_1, x_2\}$ and $\{x_1, x_2\} \subseteq X$ and so

$$X \subseteq V = \{x_1, x_2\} \subseteq X$$

from which it results that $X = V$ and condition $V - X \neq \oslash$ is not valid.

Case 2. $|V_1| \neq 1$ and $|V_2| \neq 1$.

Because G_1 is a component of G, G_1 as independent graph is a connected graph and so

$$X_1 = \{x \in V_1 | \omega(x) = 0\} = \oslash.$$

Because G_2 is a component in G, G_2 as independent graph is a connected graph and so

$$X_2 = \{x \in V_2 | \omega(x) = 0\} = \oslash.$$

Because $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \oslash$, it follows that

$$X = X_1 \cup X_2 = \oslash \cup \oslash = \oslash$$

and so $G = H$ and relation $CMC_G = CMC_H$ is obvious.

Case 3. $|V_1| = 1$ and $|V_2| \neq 1$.

Because $|V_1| = 1$, $V_1 = \{x_1\}$ with $\omega(x_1) = 0$ and so

$$X_1 = \{x \in V_1 | \omega(x) = 0\} = \{x_1\} .$$

Because G_2 is a component in G, G_2 as independent graph is a connected graph and so

$$X_2 = \{x \in V_2 | \omega(x) = 0\} = \oslash.$$

Because $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \oslash$, it follows that

$$X = X_1 \cup X_2 = V_1 \cup \oslash = V_1$$

and so $H = G_2$.

We have, G has a component minimal complete subgraph if and only if G_2 has a component minimal subgraph if and only if H has a component minimal complete subgraph. So if G has a component minimal complete subgraph, this subgraph of G is subgraph of H and it follows that $CMC_G \subseteq CMC_H$. Because we have also $CMC_H \subseteq CMC_G$, it results that $CMC_G = CMC_H$.

Case 4. $|V_1| \neq 1$ and $|V_2| = 1$.

This case is similar with case 3 with V_1 replaced by V_2 and V_2 replaced by V_1.

Note 2.2. *Let $G = (V, E)$ be a tree. Then G has not a component minimal complete subgraph.*

Note 2.2 is obvious true because if G has a component minimal complete subgraph, this subgraph contains a loop and so G contains a loop and this is a contradiction with definition of trees.

Condition 2.2. *Let $G = (V, E)$ be a graph with components $G_1, G_2, \ldots, G_k$. If for any $i = 1, 2, \ldots, k$ G_i is a tree then the determination of component minimal complete subgraph of G has no solution.*

Proposition 2.2. *Let $G = (V, E)$ be a graph with components $G_1, G_2, \ldots, G_k$. We consider the set*

$$I = \{i | G_i \text{ is a tree}\} ,$$

$$X = \bigcup_{i \in I} V_i$$

and the subgraph $H = (V - X, F)$ of G. Then, G has a component minimal complete subgraph if and only if H has a component minimal complete subgraph.

In addition, we have

$$CMC_G = CMC_H.$$

Proofs are similar with those for proposition 2.1.

Propositions 2.1 and 2.2 allow us to eliminate from a graph G all isolated vertices and all tree components when we explore the graph for its component minimal complete subgraph.

To present the next result we must give the following definition:

Definition 2.1 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \oslash$. If $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, then the graph $G = (V, E)$ is **reunion graph** of G_1 and G_2 and we write*

$$G = G_1 \cup G_2.$$

We can generalize definition 2.1 to a finite number of graphs $G_i = (V_i, E_i)$ with $i = 1, 2, \ldots, n$ for which $V_i \cap V_j = \oslash$ for any $i \neq j$, $i, j = 1, 2, \ldots, n$. We now can consider the sets

$$V = \bigcup_{i=1}^{n} V_i = \bigcup_{1 \leq i \leq n} V_i$$

and

$$E = \bigcup_{i=1}^{n} E_i$$

and the graph $G = (V, E)$ is the reunion of finite family of graphs $G_1, G_2, \ldots, G_n$ and we write

$$G = \bigcup_{i=1}^{n} G_i.$$

Now let consider a graph G without isolated vertices and without tree components. We can give the following result:

Proposition 2.3. *Let $G = (V, E)$ be a graph without isolated vertices and without tree components. We consider that $G_1, G_2, \ldots, G_k$ are the components of G. If there exist i, $1 \leq i \leq k$ for which G_i has a component minimal complete subgraph, then G has a component minimal complete subgraph and if*

$$J = \{i | \textit{it exists } CMC_{G_i} \textit{ for } G_i\}$$

then

$$CMC_G = \bigcup_{i \in J} CMC_{G_i}.$$

Proofs. If $G_i = (V_i, E_i)$ has a component minimal complete subgraph H_i, then H_i is a subgraph of G and is also a component minimal complete subgraph of G.

If G_i has a component minimal complete subgraph, namely CMC_{G_i} and G_j has a component minimal complete subgraph, namely CMC_{G_j}, because for

$i \neq j$, $V_i \cap V_j = \oslash$, it follows that $G_i \cup G_j$ has a component minimal complete subgraph, namely $CMC_{G_i \cup G_j}$ and

$$CMC_{G_i \cup G_j} = CMC_{G_i} \cup CMC_{G_j}.$$

If G_i has a component minimal complete subgraph, namely CMC_{G_i} and G_j has not a component minimal complete subgraph, then $G_i \cup G_j$ has the same component minimal complete subgraph as G_i and so

$$CMC_{G_i \cup G_j} = CMC_{G_i}.$$

By generalization it follows that proposition 2.3 is true.

Proposition 2.3 allows us to reduce our study on a connected graph with al least two vertices which is not a tree.

3. Elimination of vertices and edges

In this section we will consider connected graphs and we will try to eliminate vertices and edges without modifying component minimal complete subgraphs.

Note 3.1. *Let $G = (V, E)$ be a minimal a minimal complete graph. For any $x \in V$,*

$$\omega(x) \geq 2.$$

From definition 1.7., because for any $x \in V$ there exists $y, z \in V$ so that $\{\{x, y\}, \{x, z\}\} \subset E$ it follows that

$$\{y, z\} \subseteq \{t \in V | \{x, t\} \in E\}$$

from which

$$2 = |\{y, z\}| \leq |\{t \in V | \{x, t\} \in E\}|$$

and so $\omega(x) \geq 2$.

Proposition 3.1. *Let $G = (V, E)$ be a connected graph and $X = \{x \in V | \omega(x) = 1\}$. Let us consider the subgraph $H = (V - X, F)$ of G. Then, G gas a component minimal complete subgraph if and only if H has a component minimal complete subgraph and*

$$CMC_G = CMC_H.$$

Proofs. If H has a component minimal complete subgraph and H is subgraph of G, it follows that component minimal complete subgraph of H is subgraph in G and so G has c component minimal complete subgraph. So we have

$$CMC_H \subseteq CMC_G.$$

Let us consider $CMC_G = (V, E)$. If $x \in V$, then $\omega(x) \geq 2$ and so $x \notin X$, what it means that $x \in V - X$. This is true for any vertices in V. Then CMC_G is a component minimal complete subgraph in H, from which we have that H has a component minimal complete subgraph and

$$CMC_G \subseteq CMC_H.$$

Because we already have $CMC_H \subseteq CMC_G$, it follows that

$$CMC_G = CMC_H.$$

Note 3.2. *Let $G = (V, E)$ be a minimal complete graph. If $x \in V$ has $\omega(x) = 2$ then there exists unique $y, z \in V$ so that*

$$\{\{x, y\}, \{x, z\}\} \subset E.$$

If note 3.2. is not true, let $u \in V$ be a vertice with $\{x, u\} \in E$ and $u \neq y$, $u \neq z$. It follows that $\omega(x) \geq 3$ which is a contradiction.

Note 3.3. *Let $G = (V, E)$ be a minimal complete graph. If $x \in V$ has $\omega(x) = 2$ and $y, z \in V$ are the vertices for which $\{\{x, y\}, \{x, z\}\} \subset E$, then $\{y, z\} \in E$.*

If note 3.3 is not true, it follows that

$$\{\{x, y\}, \{x, z\}, \{y, z\}\} \cap E = \{\{x, y\}, \{x, z\}\}$$

and so the condition given in definition 1.7. is false. So G is not a minimal complete graph which is a contradiction.

Proposition 3.2. *Let $G = (V, E)$ be a connected graph and*

$$X = \{x \in V | \omega(x) = 2, \ \{x, y\}, \{x, z\} \in E \text{ and } \{y, z\} \notin E\}.$$

Let consider the subgraph $H = (V - X, F)$ of G. Then, G has a component minimal complete subgraph if and only if H has a component minimal complete subgraph and

$$CMC_G = CMC_H.$$

Proofs are similar with those for proposition 3.1.

Note 3.4. *Let $G = (V, E)$ be a minimal complete graph. Then it does not exists $x, y \in V$ with $\{x, y\} \in E$ so that partial graph $H = (V, E - \{\{x, y\}\})$ of G is a disconnected graph.*

Let us consider that there exists $x, y \in V$ with $\{x, y\} \in E$ so that partial graph $H = (V, E - \{\{x, y\}\})$ of G is a disconnected graph.

Because H is a disconnected graph it follows that it exists V_1 and V_2 so that $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \oslash$ and if

$$E_i = \{\{u, v\} \in E - \{\{x, y\}\} \, | u, v \in V_i\}$$

for $i = 1, 2$ then (V_1, E_1) and (V_2, E_2) are components of H. We can assume that $x \in V_1$ and $y \in V_2$.

If $u \in V_1$ with $u, x \in E_1$, then because V_1 and V_2 are components of H, it means that $\{u, y\} \notin E$ and so

$$\{\{x, y\}, \{x, u\}, \{y, u\}\} \cap E = \{\{x, y\}, \{x, u\}\}$$

which is in contradiction with definition 1.7.

If $u \in V_2$ with $\{u, y\} \in E_2$, then because V_1 and V_2 are components of H, it means that $\{u, x\} \notin E$ and so

$$\{\{x, y\}, \{x, u\}, \{y, u\}\} \cap E = \{\{x, y\}, \{y, u\}\}$$

which is in contradiction with definition 1.7.

Proposition 3.3. *Let $G = (V, E)$ be a connected graph and*

$$T = \{u \in E | \bar{G} = (V, E - \{u\}) \textit{ is disconnected}\}.$$

We consider partial graph $\bar{G} = (V, E - T)$ of G. Let

$$X = \{x \in V | \omega(x) = 0 \textit{ in } \bar{G}\}$$

and the subgraph $H = (V - X, F)$ of $\bar{G}$. Then, G has a component minimal complete subgraph if and only if H has a component minimal complete subgraph and

$$CMC_G = CMC_H.$$

Proofs. If H has a component minimal complete subgraph, because H is subgraph in $\bar{G}$, it results that $\bar{G}$ has a component minimal complete subgraph and $CMC_H \subseteq CMC_{\bar{G}}$. But $\bar{G}$ is disconnected and it is a partial graph of G and we can apply proposition 2.3 and note 3.4 and so if $\bar{G}$ has a component minimal complete subgraph, then G has a component minimal complete subgraph.

Because any $u \in T$ is not in a component minimal complete subgraph of G, then $CMC_{\bar{G}} = CMC_G$ and so $CMC_H \subseteq CMC_G$.

We have already shown that $CMC_{\bar{G}} = CMC_G$ and the proofs for

$$CMC_{\bar{G}} \subseteq CMC_H$$

is similar with those in proposition 3.1.

4. **Conclusion**

The subject of this paper is dedicated to some properties which allow us to eliminate edges and vertexes. This is done for determine a subgraph in which any vertex is in at least a complete graph formed with three vertexes.

The properties I present here could generate an algorithm for which, in worse case, a vertex or an edge is eliminated and so has the complexity equals with $n \times m$, where n is the number of vertexes and m is the number of edges.

I do not consider here this algorithm. In a future paper I wish to specify algorithms to solve this problem, in terms of algebra and in terms of sets operation.

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