GENERALIZED QUASI-EINSTEIN WARPED PRODUCTS

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Abstract

In this paper we investigate when an warped product manifold is a generalized quasi-Einstein manifold and we give the expressions of the Ricci tensors and scalar curvatures for the bases and fibres. In some cases we give some obstructions to the existence of such manifolds.

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1. **Introduction**

In this paper we will investigate when an warped product manifold is a generalized quasi Einstein manifold. In ([4]) it is introduced the notion of quasi-Einstein manifold.

Definition 1.1. A non-flat Riemannian manifold (M^n, g), $n > 2$ is said to be a *quasi Einstein* manifold if its Ricci tensor Ric_M of type $(0,2)$ is not identically zero and satisfies the condition $Ric_M(X,Y) = ag(X,Y) + bA(X)A(Y)$ for every $X, Y \in \Gamma(TM)$ where a, b are real scalars and A is a non-zero 1-form on M such that $A(X) = g(X,U)$ for all vector field $X \in \Gamma(TM)$, U being an unit vector field which is called the generator of the manifold. If $b = 0$, then the manifold reduces to an Einstein manifold.

In ([5]) it is given a generalization of the notion of quasi-Einstein manifold.

Definition 1.2. A non-flat Riemannian manifold (M^n, g), $n > 2$ is called a *generalized quasi-Einstein* manifold if its Ricci tensor Ric_M of type $(0,2)$ is non-zero and satisfies the condition $Ric_M(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y)$ for every $X, Y \in \Gamma(TM)$ where a, b, c are real scalars and A, B two non-zero 1-forms. The unit vector fields U and V corresponding to the 1-forms A and respectively B are defined by $A(X) = g(X,U)$, $B(X) = g(X,V)$ and are orthogonal, i.e. $g(U,V) = 0$. If $c = 0$, then the manifold reduces to a quasi Einstein manifold.

In ([1]) it is defined the notion of warped product.

Definition 1.3. Let $(B, g_B), (F, g_F)$ be two Riemannian manifolds with $\dim B = m > 1$, $\dim F = k > 0$ and $f : B \longrightarrow (0, \infty)$, $f \in C^\infty(B)$. The *warped* *product* $M = B \times_f F$ is the Riemannian manifold $B \times F$ furnished with the metric $g = g_B + f^2 g_F$. B is called the *base* of M, F the *fibre* and the warped product is called a simply Riemannian product if f is a constant function. We denote by Ric_B, Ric_F and H^f the lifts to M of the Ricci curvatures of B and F, and the $Hessian$ of f, respectively.

We give the Ricci curvature of an warped product.

Proposition 1.1. *([8]) The Ricci curvature Ric_M of the warped product $M = B \times_f F$ satisfies*:

(1) $Ric_M(X,Y) = Ric_B(X,Y) - \frac{k}{f} H^f(X,Y)$,

(2) $Ric_M(X,V) = 0$,

(3) $Ric_M(V,W) = Ric_F(V,W) - g(V,W)f^{\#}$, *where* $f^{\#} = -\frac{\triangle f}{f} + \frac{k-1}{f^2} | \bigtriangledown f|^2$ *for any vectors* $X, Y \in \Gamma(TB)$ *and any vectors* $V, W \in \Gamma(TF)$, *where* H^f *and* $\triangle f$ *denote the Hessian of* f *and the Laplacian of* f *given by* $\triangle f = -Tr(H^f)$.

We give now the scalar curvature of an warped product.

Proposition 1.2. *([1]) Let $M = B \times_f F$ be an warped product. Then the scalar curvature of M is given by* $\tau_M = \tau_B + \frac{\tau_F}{f^2} + 2k\frac{\triangle f}{f} - k(k-1)\frac{|\nabla f|^2}{f^2}$, where τ_B and τ_F are the scalar curvature of B and F, respectively.

2. **Generalized quasi-Einstein warped products**

Let $M = B \times_f F$ be an warped product manifold with $f : B \longrightarrow (0, \infty)$, $f \in C^\infty(B)$ and the metric $g = g_B + f^2 g_F$ which is also a generalized quasi-Einstein manifold. That means its Ricci tensor satisfies

$$Ric_M(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \quad (1)$$

We can also write the Ricci tensor as $Ric_M(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) = ag(X,Y) + bg(X,U)g(Y,U) + cg(X,V)g(Y,V)$ because $A(X) = g(X,U)$, $B(X) = g(X,V)$ for every $X, Y \in \Gamma(TM)$ with U, V unitary vector fields such that $g(U,V) = 0$. We want to compute the Ricci tensors of B and F. For that we will consider the following cases: a). when U, V are tangent to B; b). when U, V are tangent to F; c). when U is tangent to B and V is tangent to F.

Theorem 2.1. *Let $M = B \times_f F$ be an warped product manifold which is also a generalized quasi-Einstein manifold with its Ricci tensor satisfying (1).*

a). When U and V are orthogonal and tangent to B the Ricci tensors of B and F satisfy the following equations:

$$\begin{cases} Ric_B(X,Y) = ag_B(X,Y) + \frac{k}{f}H^f(X,Y) + bg_B(X,U)g_B(Y,U) + \\ \qquad cg_B(X,V)g_B(Y,V), \\ Ric_F(X,Y) = g_F(X,Y)[-f\triangle f + (k-1)| \bigtriangledown f|^2 + af^2]. \end{cases} \quad (2)$$

b). *When U and V are orthogonal and tangent to F the Ricci tensors of B and F satisfy the following equations:*

$$\begin{cases} Ric_B(X,Y) = ag_B(X,Y) + \frac{k}{f}H^f(X,Y) \\ Ric_F(X,Y) = g_F(X,Y)[-f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2]+ \\ \qquad\qquad bf^4g_F(X,U)g_F(Y,U) + cf^4g_F(X,V)g_F(Y,V) \end{cases} \tag{3}$$

c). *When U is tangent to B and V is tangent to F the Ricci tensors of B and F satisfy the following equations:*

$$\begin{cases} Ric_B(X,Y) = ag_B(X,Y) + \frac{k}{f}H^f(X,Y) + bg_B(X,U)g_B(Y,U). \\ Ric_F(X,Y) = g_F(X,Y)[-f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2]+ \\ \qquad\qquad cf^4g_F(X,V)g_F(Y,V) \end{cases} \tag{4}$$

Proof. a). For $X,Y \in \Gamma(TB)$ we have that $Ric_M(X,Y) = ag_B(X,Y)+ bg_B(X,U)g_B(Y,U)+ cg_B(X,V)g_B(Y,V)$. Hence from proposition 1.1. we have $Ric_M(X,Y) = Ric_B(X,Y) - \frac{k}{f}H^f(X,Y)$. For $X,Y \in \Gamma(TF)$ we have that $Ric_M(X,Y) = af^2g_F(X,Y)$. Thus from proposition 1.1. we have $Ric_M(X,Y) = Ric_F(X,Y) - f^2g_F(X,Y)[-\frac{\triangle f}{f} + \frac{k-1}{f^2}|\bigtriangledown f|^2]$ and so we obtain the result.

b). For $X,Y \in \Gamma(TB)$ we have that $Ric_M(X,Y) = ag_B(X,Y)$. Hence from proposition 1.1. we have $Ric_M(X,Y) = Ric_B(X,Y) - \frac{k}{f}H^f(X,Y)$. For $X,Y \in \Gamma(TF)$ we have that $Ric_M(X,Y) = af^2g_F(X,Y)+ bf^4g_F(X,U)g_F(Y,U)+ cf^4g_F(X,V)g_F(Y,V)$. Thus from proposition 1.1. we have $Ric_M(X,Y) = Ric_F(X,Y) - f^2g_F(X,Y)[-\frac{\triangle f}{f} + \frac{k-1}{f^2}|\bigtriangledown f|^2]$ and so we have the result.

c). For $X,Y \in \Gamma(TB)$ we have that $Ric_M(X,Y) = ag_B(X,Y)+bg_B(X,U)g_B(Y,U)$. Hence from proposition 1.1. we have $Ric_M(X,Y) = Ric_B(X,Y) - \frac{k}{f}H^f(X,Y)$. For $X,Y \in \Gamma(TF)$ we have that $Ric_M(X,Y) = af^2g_F(X,Y)+cf^4g_F(X,V)g_F(Y,V)$. Thus from proposition 1.1. we have $Ric_M(X,Y) = Ric_F(X,Y) - f^2g_F(X,Y)[-\frac{\triangle f}{f} + \frac{k-1}{f^2}|\bigtriangledown f|^2]$ and so we have the result.

We can give now the scalar curvatures of M, B and F.

Corollary 2.1. *Taking the traces in theorem 2.1., point a). we obtain:*

$$\begin{cases} \tau_M = (m+k)a + b + c \\ \tau_B = ma - k\frac{\triangle f}{f} + b + c \\ \tau_F = k[-f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2] \end{cases} \tag{5}$$

Corollary 2.2. *Taking the traces in theorem 2.1., point b). we obtain:*

$$\begin{cases} \tau_M = (m+k)a + b + c \\ \tau_B = ma - k\frac{\triangle f}{f} \\ \tau_F = k[-f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2]+ \\ \qquad (b+c)f^4 \end{cases} \tag{6}$$

Corollary 2.3. *Taking the traces in theorem 2.1., point c). we obtain:*

$$\begin{cases} \tau_M = (m+k)a + b + c \\ \tau_B = ma - k\frac{\triangle f}{f} + b \\ \tau_F = k[-f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2] + cf^4 \end{cases} \qquad (7)$$

3. **Obstructions to the existence of generalized quasi-Einstein warped products**

In this section we prove some obstructions to the existence of generalized quasi-Einstein warped products. We consider three cases, depending on U, V tangent to the base B or to the fibre F.

1). When U, V are tangent to F.

Theorem 3.1. *Let $M = B \times_f F$ be an warped product with* $\dim B = m \geq 2$, $\dim F = k \geq 1$ *which is also a generalized quasi-Einstein manifold with $Ric_M(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y)$, $a, b, c \in \mathbb{R}$, $A(X) = g(X,U)$, $B(X) = g(X,V)$ for every $X, Y \in \Gamma(TM)$ with U, V an unitary and orthogonal vector fields. If $b \neq 0$ or $c \neq 0$, then M reduces to a simply Riemannian product.*

Proof: Consider in the second equation of (3) that X, Y are orthogonal vectod fields tangent to F such that $g(X,U) \neq 0$ and $g(Y,U) \neq 0$ or $g(X,V) \neq 0$ and $g(Y,V) \neq 0$. Then taking in consideration the different domains of definition of the functions that apper in the second equation of (3), we obtain that f is constant.

2). When U is tangent to B and V is tangent to F.

Theorem 3.2. *Let $M = B \times_f F$ be an warped product with* $\dim B = m \geq 2$, $\dim F = k \geq 1$ *which is also a generalized quasi-Einstein manifold with $Ric_M(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y)$, $a, b, c \in \mathbb{R}$, $A(X) = g(X,U)$, $B(X) = g(X,V)$ for every $X, Y \in \Gamma(TM)$ with U, V an unitary and orthogonal vector fields. If $c \neq 0$, then M reduces to a simply Riemannian product.*

Proof: Consider in the second equation of (3) that X, Y are orthogonal vectod fields tangent to F such that $g(X,V) \neq 0$ and $g(Y,V) \neq 0$. Then taking in consideration the different domains of definition of the functions that apper in the second equation of (3), we obtain that f is constant.

3). When U, V are tangent to B.

Remark 3.1. From now on we consider in the second equation of (2) that:

$$-f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2 = \gamma \in \mathbb{R} \text{ constant} \qquad (8)$$

Remark 3.2. Thus the Ricci tensors from (2) become:

$$\begin{cases} Ric_B(X,Y) = ag_B(X,Y) + \frac{k}{f}H^f(X,Y) + \\ \qquad\qquad bg_B(X,U)g_B(Y,U) + cg_B(X,V)g_B(Y,V), \\ Ric_F(X,Y) = \gamma g_F(X,Y), \\ -f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2 = \gamma \end{cases} \qquad (9)$$

Theorem 3.3. *Let $M = B \times_f F$ be an warped product with B compact and connected, $\dim B = m \geq 2$, $\dim F = k \geq 2$ which is also a generalized quasi-Einstein manifold with $Ric_M(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y)$, $a,b,c \in \mathbb{R}$, $A(X) = g(X,U)$, $B(X) = g(X,V)$ for every $X,Y \in \Gamma(TM)$ with U,V* an unitary orthogonal vector fields. *If $a \leq 0$, then M reduces to a simply Riemannian product.*

Proof: Let $z \in N$ such that $f(z)$ is the maximum of f on N. Then $\bigtriangledown f(z) = 0$ and $\triangle f(z) \geq 0$. Writing the equation (8) in the point z we obtain:

$$-f(z)\triangle f(z) + af^2(z) = \gamma \qquad (10)$$

Now from (8) and (10) we obtain that:

$$\begin{gathered} -f(z)\triangle f(z) + af^2(z) = -f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2 \Longrightarrow \\ f\triangle f = (k-1)|\bigtriangledown f|^2 + af^2 + f(z)\triangle f(z) - af^2(z) \Longrightarrow \\ f\triangle f = (k-1)|\bigtriangledown f|^2 + f(z)\triangle f(z) + a[f^2 - f^2(z)] \geq 0 \Longrightarrow \triangle f \geq 0 \end{gathered}$$

Thus f is constant.

From now on we will consider that $a > 0$.

Theorem 3.4. *Let $M = B \times_f F$ be an warped product with B compact and connected, $\dim B = m \geq 2$, $\dim F = k \geq 2$ which is also a generalized quasi-Einstein manifold with $Ric_M(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y)$, $a,b,c \in \mathbb{R}$, $A(X) = g(X,U)$, $B(X) = g(X,V)$ for every $X,Y \in \Gamma(TM)$ with U,V an unitary orthogonal vector fields. If F has negative scalar curvature, then M reduces to a simply Riemannian product.*

Proof: From the third equation of (4) and (8) we get that $\tau_F = k\gamma$. Since $\tau_F \leq 0$ it follows that $\gamma \leq 0$. Then (8) becomes:

$$\begin{gathered} -f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2 = \gamma \Longrightarrow \\ -f\triangle f + af^2 = \gamma - (k-1)|\bigtriangledown f|^2 \leq 0 \Longrightarrow \\ f\triangle f \geq af^2 > 0 \Longrightarrow \triangle f > 0 \end{gathered}$$

Thus f is constant.

From now on we will also consider that $\gamma > 0$.

Theorem 3.5. *Let $M = B \times_f F$ be an warped product with B compact and connected, $\dim B = m \geq 2$, $\dim F = k \geq 2$ which is also a generalized quasi-Einstein manifold with $Ric_M(X,Y) = ag(X,Y) + bA(X)A(Y) +$*

$cB(X)B(Y)$, $a, b, c \in \mathbb{R}$, $A(X) = g(X, U)$, $B(X) = g(X, V)$ for every $X, Y \in \Gamma(TM)$ with U, V an unitary orthogonal vector fields. If one of the following condition is true, then M reduces to a simply Riemannian product:

a). $|\bigtriangledown f|^2 \geq \frac{\gamma}{k-1}$,
b). $\tau_M \leq \tau_B$,
c). $\tau_M \geq \tau_B + \tau_F$ and $\gamma \geq a$,
d). $\tau_B \leq 0$ and $b + c \geq 0$,
e). $\tau_B \geq \frac{m\gamma}{f^2} + b + c$ and $m \geq k$,

Proof: a). From (8) we have that:

$$\begin{gathered} -f\triangle f + (k-1)|\bigtriangledown f|^2 + af^2 = \gamma \Longrightarrow \\ -f\triangle f + af^2 = \gamma - (k-1)|\bigtriangledown f|^2 \leq 0 \Longrightarrow \\ f\triangle f \geq af^2 \Longrightarrow \triangle f \geq 0. \end{gathered}$$

Thus f is constant.

b). From the second equation of (4) we have that:

$$\begin{gathered} \tau_B = ma - k\frac{\triangle f}{f} + b \Longrightarrow \\ \tau_B + ak = a(m+k) - k\frac{\triangle f}{f} + b = \tau_M - k\frac{\triangle f}{f} \Longrightarrow \\ \tau_M - \tau_B = ak + k\frac{\triangle f}{f} = \frac{k}{f}[af + \triangle f] \leq 0 \Longrightarrow \\ \triangle f \leq -af < 0. \end{gathered}$$

Thus f is constant.

c). From the third equation of (4) and (8) we get that:

$$\begin{gathered} \tau_F = k\gamma \geq ka \Longrightarrow \\ \tau_F + \tau_B \geq ka + ma - k\frac{\triangle f}{f} + b = \tau_M - k\frac{\triangle f}{f} \Longrightarrow \\ k\frac{\triangle f}{f} \geq \tau_M - (\tau_F + \tau_B) \geq 0 \Longrightarrow \\ \triangle f \geq 0. \end{gathered}$$

Thus f is constant.

d). From the second equation of (4) we have that:

$$\begin{gathered} \tau_B = ma - k\frac{\triangle f}{f} + b + c \Longrightarrow \\ k\frac{\triangle f}{f} = ma - \tau_B + b + c \geq 0 \Longrightarrow \\ \triangle f \geq 0. \end{gathered}$$

Thus f is constant.

e). From the second equation of (4) and (8) we have that:

$$\tau_B = ma - k\frac{\triangle f}{f} + b + c \Longrightarrow$$
$$\tau_B f^2 = maf^2 - kf\triangle f + (b+c)f^2 =$$
$$m(\gamma + f\triangle f - (k-1)|\bigtriangledown f|^2) - kf\triangle f + (b+c)f^2 =$$
$$(m-k)f\triangle f + m\gamma - m(k-1)|\bigtriangledown f|^2 + (b+c)f^2 \Longrightarrow$$
$$\tau_B f^2 - m\gamma - (b+c)f^2 = (m-k)f\triangle f - m(k-1)|\bigtriangledown f|^2 \geq 0.$$

i). $m = k$ implies $-m(m-1)|\bigtriangledown f|^2 \geq 0 \Longrightarrow m(m-1)|\bigtriangledown f|^2 \leq 0 \Longrightarrow |\bigtriangledown f|^2 = 0 \Longrightarrow \bigtriangledown f = 0$.

ii). $m > k$ implies $(m-k)f \triangle f \geq m(k-1)|\bigtriangledown f|^2 \geq 0 \Longrightarrow \triangle f \geq 0$.

Thus f is constant.

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