

STIFF EQUATION WITH APPLICATION IN PHYSIOLOGICAL EXCITABLE SYSTEMS

TURNEA, Marius

ILEA, Mihai

ROTARIU, Mariana

University of Medicine and Pharmacy "GR.T.Popa",

Faculty of Medical Bioengineering,

Iasi, Romania

marius.turnea@bioinginerie.ro, ileamihai2004@yahoo.com,

rotariu29@yahoo.com

Abstract

The Belousov-Zhabotinsky reaction is quite a fascinating scientific phenomenon such that its cyclical nature can be seen as analogues to: the heartbeat, circadian rhythms, the menstrual cycle, variations in hormone levels, and many others. A nerve cell is thus an example of an excitable medium: the nerve cell returns to its resting steady state after small stimuli, but a sufficiently large stimulus generates an activation before the nerve cell returns to equilibrium. This reaction is a frequently used example for an excitable medium showing propagating oxidation waves. When travelling in a liquid solution without a gel, the chemical waves induce a convective flow in the reaction medium. We will specifically consider the situations in which chemical formalism explain insufficient the formation of biological pattern. Chemical system is also useful to demonstrate the development of the mathematical apparatus which is routinely used in all areas of pattern formation.

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1. Introduction

Mathematical models of chemical systems are often formulated as a large set of differential equations with small parameter, and for purpose of numerical and/or analytical analysis it is often desirable to reduce to a smaller system without essential loss of accuracy. When several small singular perturbation parameters of the same order of magnitude are present in the dynamic model of a physical system, the control problem is usually solved as a single parameter perturbation problem, such a system is called a singularly perturbed system.

Certain chemical reactions are able to oscillate in time and in space. The modelling of many chemical systems calls for high-order dynamic equations. The presence of some parameter is often the source for the increased order and "stiffness" of these systems. The stiffness attributed to the simultaneous occurrence of "slow" and "fast" phenomena, gives rise to time scales. The oscillating chemical reaction exists throughout nature. A Belousov-Zhabotinsky reaction (Tyson, 1985) is one of a class of reactions that serve as a classical example of non-equilibrium thermodynamics, resulting in the establishment of a nonlinear chemical oscillator. This reaction is an example of a temporally oscillating chemical reaction. The only common element in these oscillating systems is the inclusion of bromine and an acid. The reactions are theoretically important in that they show that chemical reactions do not have to be dominated by equilibrium thermodynamic behaviour. These reactions are far from equilibrium and remain so for a significant length of time. In this sense, they provide an interesting chemical model of non equilibrium biological phenomena, and the mathematical models of the Belousov-Zhabotinsky reactions themselves are of theoretical interest. The Belousov-Zhabotinsky (BZ) reaction is a family of oscillating chemical reactions. During these reactions, transition-metal ions catalyze oxidation of various, usually organic, resultants by bromes acid in acidic water solution. Most (BZ) reactions are homogeneous. The (BZ) reaction makes it possible to observe development of complex patterns in time and space by naked eye on a very convenient human time scale of dozens of seconds and space scale of several millimetres.

The (BZ) reaction can generate up to several thousand oscillatory cycles in a closed system, which permits studying chemical waves and patterns without constant replenishment of reactants. The rates of this reaction are worked out from the reaction and it is usually dependent on the concentration of the reacting molecules or sometimes on the concentration of the molecules that are formed from this reaction.

2. Material and Methods

We present a differential system with small parameter for non physiological system presented by (Tyson, 1980) for the two-variable model. There are three main processes, which are based on this model. They are bromide consumption (A), the autocatalytic stage (B), and regeneration stage (C). The Bromide transforms into different oxidized forms, which is process A . The role of this process is to remove them Br^- . This ion is the inhibitor for process B , the autocatalytic stage. Process B only starts when the bromide ion reaches a lower critical value, whilst the $HBrO_2$ (activator). The oxidation of the catalyst transforms Ce^{3+} . Into Ce^{4+} Kinetic equations describing this reaction are: x denotes the concentration of bromides acid, and y denotes the concentration of the oxidized catalyst metal. This reaction is name reaction Belousov-Zhabotinsky. The system of ordinary differential equations

with small parameter involves coupled slow and fast motions of the form:

$$\begin{cases} \varepsilon \frac{dx}{dt} = -y \frac{x-a}{x+a} + x - x^2 \\ \frac{dy}{dt} = x - y \end{cases} \quad (1)$$

The initial value problem is determined by the initial conditions:

$$x(0) = x_0, \quad y(0) = y_0 \quad (2)$$

3. Results

The choice of a small parameter is connected with the order of the solution and the asymptotic sequence which describes the behaviour of the solution as $\varepsilon \rightarrow 0$. The fact that a small parameter ε multiplies a derivative in (1) indicates that it is a singular perturbation problem. One class of such problems is immediately recognised if, on setting $\varepsilon = 0$, the order of the system of differential equations is reduced, such a reduced system cannot in general satisfy all the initial condition (2).

The solution to (1) depends on the parameter $\varepsilon > 0$. The variables y and x are referred to as the slow state and the fast state respectively. The standard approach examines conditions guarantee that the solutions of (1) converge, as $\varepsilon \rightarrow 0$, to the solution of the differential algebraic system, obtained when in (1) the value of the parameter is set as $\varepsilon \rightarrow 0$ (De Jager, 1996). Settings $\varepsilon = 0$ in (1) yields the limit of solutions as $\varepsilon \rightarrow 0$ under restrictive conditions. In particular, a crucial condition is that for each fixed y , solutions of the differential equations:

$$\frac{dy}{ds} = -y \frac{x-a}{x+a} + x - x^2 \quad (3)$$

Should converge, as $s \rightarrow \infty$ to a solution of the algebraic equation:

$$-y \frac{x-a}{x+a} + x - x^2 = 0 \quad (4)$$

Where (4) is a "cubic" shape. Note that $a \sim 10^{-4}$ and $\varepsilon > 0$ (generally taken to be small). By this we mean that for a finite range of value for y , there are three solutions $x = x(y)$ of the equation (4). These we will denote by:

$$x = X_0(y), x = X_1(y), x = X_2(y) \quad (5)$$

and where comparison is possible:

$$X_0(y) \leq X_1(y) \leq X_2(y) \quad (6)$$

We denote the minimal value of y for which $X_0(y)$ exists by Y_1 and the maximal value of y for which $X_2(y)$ exists by Y_2 . The equation:

$$x - y = 0 \quad (7)$$

Is assumed to have precisely one interaction with the curbed determined by (4). Increasing x beyond the curbed $f(x, y) = x - y = 0$ makes $x > y$ and decreasing y below the curbed $f(x, y) = 0$.

$$\begin{cases} x = y \\ -y \frac{x-a}{x+a} + x - x^2 = 0 \end{cases} \quad (8)$$

$$\begin{cases} x = y \\ x(2a - x^2 - ax) = 0 \end{cases} \quad (9)$$

The fast variable x is called the excitation variable, while the slow variable y is called the recovery variable. From relations (9) you have:

$$y = \frac{(x - x^2)(x - a)}{x + a} \quad (10)$$

This means that if possible, x is adjusted rapidly to maintain pseudo-equilibrium at equation (4). In other words, if possible, x clings to the stable branches of equation (4) namely $x = X_0(y)$ and $x = X_2(y)$. Along these branches the dynamics of y are governed by the reduced dynamics (Tyson, 1988):

$$\begin{cases} \frac{dy}{dt} = -y \frac{X_0(y)-a}{X_0(y)+a} + X_0(y) - X_0^2(y) \\ \frac{dy}{dt} = -y \frac{X_2(y)-a}{X_2(y)+a} + X_2(y) - X_2^2(y) \end{cases} \quad (11)$$

When it is not possible for x to be in quasi-equilibrium, the motion is governed approximately by the differential equations:

$$\begin{cases} \frac{dx}{d\pi} = -y \frac{x-a}{x+a} + x - x^2 \\ \frac{dy}{d\pi} = 0 \end{cases} \quad (12)$$

Found by making the change of variables to the fast time scale $\pi = \varepsilon t$, and then setting $\varepsilon = 0$. On this time scale, y is constant, while x equilibrates to a stable solution of (4). By assumption, there is only one steady at $x = \tilde{x}$, $y = \tilde{y}$ with:

$$\begin{cases} \tilde{x} = \tilde{y} \\ -\tilde{y} \frac{\tilde{x}-a}{\tilde{x}+a} + \tilde{x} - \tilde{x}^2 = 0 \end{cases} \quad (13)$$

Without loss of generality, we may assume that this ready state occurs at the origin, as this involves a shift of the variables (Keener, 1998). That is, if parameters are varied so that the steady state passes through this point, a periodic orbit arises as a continuous solution branch and bifurcates into a stable cycle oscillation. Presence of this small parameter is for indicating a phenomenon scaling according of time. This allows reducing this system of equations explicitly by choosing parameters. Smallness of parameter ε indicates that the time dynamics of x is faster than that y . When the steady state is on the left most branches, but close to the minimum, the system is excitable. This is because even though the steady state is linearly stable, a sufficiently large perturbation from the steady state sends the state variable on a trajectory that runs away from the steady state before eventually returning

to rest. The curve $x = X_1(y)$ is a threshold curve. While x remains on the upper branch, y increases:

$$\frac{dy}{dt} = -y \frac{X_2(y) - a}{X_2(y) + a} + X_2(y) - X_2^2(y) \quad (14)$$

In the finite time:

$$T = \int_{y_0}^{y_1} \frac{(X_2(y) + a)dy}{-X_2^3(y) + X_2^2(y)(1 - a) + X_2(y)(a - y) + ay} \quad (15)$$

This period of time constitutes the excited phase of the action potential. When y reaches y_1 it is not longer possible for x to stay on the excited branch, so it must return to the lower branch $X_0(y)$. Once on this branch, y decreases following:

$$\frac{dy}{dt} = -y \frac{X_0(y) - a}{X_0(y) + a} + X_0(y) - X_0^2(y) \quad (16)$$

If the rest point lies on the lower branch then:

$$-y \frac{X_0(\tilde{x}) - a}{X_0(\tilde{x}) + a} + X_0(\tilde{x}) - X_0^2(\tilde{x}) = 0 \quad (17)$$

And y gradually returns to rest on the lower branch.

4. Conclusions

We have examined important chemical mechanisms whose evolution can be treated as two-variable dynamical systems in the phase plane. The presence of some parameter is often the source for the increased order and "stiffness" of these systems. The stiffness attributed to the simultaneous occurrence of "slow" and "fast" phenomena, gives rise to time scales. The systems in which the suppressions of a small parameter is responsible for the reduction of order of the system are labelled as singularly perturbed systems, which are a special representation of the general class of time-scale systems. Whatever small ε we take, from a transparent and exact solution of the simple model at $\varepsilon = 0$, we abruptly face a far more complicated solution of a model that is just a little bit better. The singularly perturbed system studied has two widely separated characteristic roots giving rise to "slow" and "fast" components in its solution. If we accept approximate solution, where the approximation is based on the inherently small modelling parameter ε , we do have the possibility to gradually increase the complexity of a model, and study small but significant effects in the most efficient way. In initial-value approach of the perturbed ODE system (P_ε), initial guesses for equation's parameters are chosen and the dynamical equations are solved numerically. As a matter of fact, our concept of the perturbation search method was inspired by the prospect of basic perturbation theory. The principle of perturbation theory is to study dynamical systems that are small perturbation of integral perturbed system.

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