NUMBER OF ISOMORPHIC GRAPHS WITH THE SAME DIAGONAL CELL TYPE WALKS MATRIX

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Abstract

Considering a graph G with diagonal cell type walks matrix, the goal is to determine the number of isomorphic graph H with G for which H has the same diagonal cell type walks matrix as G. For this operation only combinatorics and graph theory (without using isomorphism properties from algebra) are used.

Firstly, some graph theory definitions used in this paper are presented. Two simple cases (equal dimensioned graph components and distinct dimensioned graph components) are presented in the first part. Then, the general case is described.

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1. **Introduction**

Firstly I wish to present some of the graph theory definitions. I make this because there are authors that used different definitions, starting with definition for graph. In such a definition, a graph G is defined as (V, E, f) where V represent the vertices, E is the set of edges and f is adjacency function.

The definition used by others (including myself) is the following:

Definition 1.1 *A **graph** G is a pair (V, E) where V is a finite set specifying the vertices (generally considered as $V = 1, 2, \ldots, n$) and E is the set of unordered pairs of numbers from V (generally presented as subsets of two values from V) named edges.*

In the above definition we do not assume that numbers from V are different so we can have x, x in E. More than that, we do not assume that if a and b are two different pairs from E, then we can not have a and b formed with the same values x and y from V.

Definition 1.2 *Let $G = (V, E)$ be a graph and x and y two different vertices from G. We have a* ***walk*** *from x to y if and only if we have a sequence*

$$x = v_1, v_2, \ldots, v_k = y$$

of vertices so that $\{v_i, v_{i+1}\} \in E$, for any $i = 1, 2, \ldots, k-1$. If $x = y$, the walk is named loop. We specify the walk from x to y by

$$x = v_1, v_2, \ldots, v_k = y.$$

Definition 1.3 *Let $G = (V, E)$ be a graph. G is a* ***connected graph*** *if and only if, for any different vertices x and y there exists a walk from x to y. If G is not connected then G is named* ***disconnected graph****.*

Definition 1.4 *Let $G = (V, E)$ be a graph and W is a subset of V. We define the subset F of E as*

$$F = \{u = \{x, y\} \in E | x, y \in W\}$$

and a new graph $H = (W, F)$. H is named ***subgraph*** *of G.*

Definition 1.5 *Let $G = (V, E)$ be a graph. A subgraph $H = (W, F)$ of G is named* ***component*** *of G if and only if H is a connected graph and for any $x \in W$ and any $y \in V - W$ does not exist an edge $\{x, y\} \in E$.*

If G is a connected graph then the only component of G is G itself. If G is a disconnected graph one can demonstrate that there exists a partition of V,

$$W_1, W_2, \ldots, W_k,$$

so that, defining

$$F_1, F_2, \ldots, F_k$$

as above for any $i = 1, 2, \ldots, k$, the subgraph $H_i = (W_i, F_i)$ is a component of G.

There are some matrices that can be associated with graphs. One of this matrix is the adjacency matrix, $A_G = (a_{ij})_{i,j=1,2,\ldots,n}$, where $a_{ij} = 1$ if and only if there exists a walk from i to j, otherwise $a_{ij} = 0$, and n represents the number of vertices. For this paper the next definition is important:

Definition 1.6 *Let $G = (V, E)$ be a graph having n vertices. We define* ***walks matrix*** *as a $0, 1$ matrix $D_G = (d_{ij})_{i,j=1,2,\ldots,n}$, where $d_{ij} = 1$ if and only if there exists a walk from i to j, otherwise $d_{ij} = 0$.*

There are algorithms to compute walks matrix D_G from the adjacency matrix A_G.

Another important definition for this paper indicates a special form for 0,1 matrix. This definition is:

Definition 1.7 *Let A be a $0, 1$ n-dimensioned matrix so that there exist numbers $n_1, n_2, \ldots, n_k$, not necessarily different, with*

$$n_1 + n_2 + \ldots + n_k = n$$

so that

$$A = \left(\begin{array}{cccc} B_1 & O & \dots & O \\ O & B_2 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & B_K \end{array} \right),$$

where for any $i = 1, 2, \dots, k$ B_i *is a* n_i*-dimensioned matrix with all elements equal* 1 *for* $n_i \neq 1$ *or* B *is zero if* $n_i = 1$*, and* O *represents matrix with all elements equal* 0*. Matrix* A *of this form is named* ***diagonal cell type matrix****.*

My aim is to solve the following problem using only combinatorics.

Problem *Giving a graph* $G = (V, E)$ *with diagonal cell type walks matrix* D*, let determine the number of graphs* $H = (V, F)$ *isomorph with* G *so that* H *has the same diagonal cell type walks matrix* D*.*

2. Isomorphic graphs

In this section I present some results related to isomorphic graphs which it's necessary to prepare my calculus. Firstly we must know what mean isomorphic graphs.

Definition 2.1 *Let* $G = (V, E)$ *and* $H = (W, F)$ *be two graphs.* G *and* H *are named isomorphic if and only if there exists a function* $f : V \to W$ *so that* f *is bijective and* $a = \{x, y\} \in E$ *if and only if* $f(a) = \{f(x), f(y)\} \in F$*.*

Firstly I must observe that V and W are finite sets and so, because f is bijective, results that V and W have the same number of elements. More than that, in first section I assumed that if $G = (V, E)$ is a graph, then for V we use $V = \{1, 2, \dots, n\}$, where n is the number of vertices. (We can consider $V = W = \{1, 2, \dots, n\}$.) In this may $f : V \to V$ is a bijective function and so f is a permutation of set V.

Note 2.1 *If* $G = (V, E)$ *and* $H = (V, F)$ *are two isomorphic graphs and* σ *is a permutation of* V *so that*

$$F = \{\{\sigma(x), \sigma(y)\} \mid \{x, y\} \in E\},$$

we write $H = \sigma(G)$*.*

If $G = (V, E)$ is a connected graph with n vertices then $D_G = B_n$ is a matrix with all elements equal 1. If $G = (V, E)$ is a connected graph and σ is a permutation of V, then $H = \sigma(G)$ is a connected graph and so $D_H = B_n$. So we have that for any permutation σ of V, $\sigma(G)$ isomorphic with G and $D_{\sigma(G)} = B_n$. It results that the number of isomorphic graph H with G for which $D_H = B_n$ is equal with the number of permutation of V and so it is $n!$. We can consider that B_n is diagonal cell type matrix (having only one cell) and so we have shown the following result:

Lemma 2.1 *Giving a connected graph* $G = (V, E)$*, the number of graphs* $H = (V, F)$ *isomorphic with* G *so that* H *has a diagonal cell type walks matrix and* $D_G = D_H$ *is* $n!$*.*

This result represents a solution for our problem when G is a connected graph.

If we consider that graph $G = (V, \oslash)$, the graph formed only with isolated vertices for which $D_G = O$ is a matrix with all elements equal with zero, then for any permutation σ we have $\sigma(G) = (V, \oslash)$ and so $D_{\sigma(G)} = O$. We have shown in this way that:

Lemma 2.2 *Giving a fully disconnected graph $G = (V, \oslash)$, the number of graphs $H = (V, F)$ isomorphic with G so that H has a diagonal cell type walks matrix and $D_G = D_H$ is $n!$.*

So our problem is to be solving for disconnected graphs. Next two sections present two particular cases, namely: equal dimension components and distinct dimension components and, when we use the term of dimension for the number of vertices

3. Graphs with equal dimension components

Let consider a disconnected graph $G = (V, E)$ with n vertices. Then there exists a partition of V with subsets $V_1, V_2, \ldots, V_k$ so that if

$$E_i = \{\{x, y\} \in E | x, y \in V_i\}$$

then $G_i = (V_i, E_i)$ is component in G. We assume that, for any $i = 1, 2, \ldots, k$, V_i has p vertices. Because $V_1, V_2, \ldots, V_k$ represent a partition of V, it results that $n = kp$.

We assume that G has a diagonal cell type walks matrix and $p \neq 1$.

Because all components in G have the same number p of vertices and G has diagonal cell type walks matrix, it follows that any cell is p-dimensioned and diagonal cell type walks matrix for G is:

$$D_G = \begin{pmatrix} B_p & O & \ldots & O \\ O & B_p & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \ldots & B_p \end{pmatrix},$$

where B_p is a p-dimensioned matrix with all elements equal 1 and O represents matrix with all elements equal 0.

In addition, values in G_i are consecutive integers.

We fix a component i and we consider a permutation σ_i of V so that restriction of σ_i to $V - V_i$ is identity permutation, and so σ_i changes values only inside of V_i.

Because σ_i is a permutation of V, $\sigma_i(G)$ is isomorphic with G. Because restriction of σ_i to $V - V_i$ is identity, it follows that in $\sigma_i(G)$ all components are unchanged, except the component corresponding to V_i. It results that the number of isomorphic graphs with G generated by permutation of type σ_i is equal with the number of permutation of V_i and so is equal with $p!$.

Because we fix independently the component i, if we generate a permutation τ of V for which $\tau(G)$ preserves the components $V_1, V_2, \ldots, V_k$ in this order, $\tau(G)$ is isomorphic with G and

$$\tau = \sigma_1 \sigma_2 \ldots \sigma_k.$$

Because of the independency in fixing component i, the number of such permutation τ is equal with $(p!)^k$.

Above we considered the order of component of G, namely $V_1, V_2, \ldots, V_k$. If we do not have this constraint, we must consider all possible orders for components. Because all components has the same number of vertices, a permutation in components order does not change the form for walks matrix and so we have $k!$ possible orders.

We showed the following result:

Lemma 3.1 *If $G = (V, E)$ is a graph with k components and all components have the same number p of vertices, then the number of isomorphic graph H with G for which H has diagonal cell type walks matrix and $D_G = D_H$ is $k!(p!)^k$.*

To complete this result we must observe that if G is a connected graph, then the number of component is $k = 1$ and number of vertices is $p = n$ and so we have $1!(n!)^1 = n!$ isomorphic graph H with G and H has diagonal cell type walks matrix and $D_G = D_H$, results that is identical with that specified by Lemma 2.1.

4. Graphs with distinct dimension components

Let's consider a disconnected graph $G = (V, E)$ with n vertices. Then there exists a partition of V with subsets $V_1, V_2, \ldots, V_k$ so that if

$$E_i = \{\{x, y\} \in E | x, y \in V_i\}$$

then $G_i = (V_i, E_i)$ is component in G for any $i = 1, 2, \ldots, k$. We assume that for any $i = 1, 2, \ldots, k$, V_i has p_i vertices and $p_i \neq p_j$ for any $1 \leq i, j \leq k$, $i \neq j$. Because $V_1, V_2, \ldots, V_k$ is a partition of V it follows that

$$n = \sum_{i=1}^{k} p_i.$$

From this assumption it results that the set $\{p_1, p_2, \ldots, p_k\}$ has k elements.

Because G has diagonal cell type walks matrix and any cell corresponds to a component in G it follows that the diagonal cell type walks matrix for G is:

$$D_G = \begin{pmatrix} B_{p_1} & O & \ldots & O \\ O & B_{p_2} & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \ldots & B_{p_k} \end{pmatrix},$$

where B_{p_i} is a p_i-dimensioned matrix with all elements equal 1 if $p_i \neq 1$ or B_{p_i} is equal with zero if $p_i = 1$ and O is a matrix with all elements equal zero.

We fix a component i and we consider a permutation σ_i of V so that $\sigma_i(j) = j$ if $j \in V - V_i$ and so σ_i changes values only inside of V_i. It follows that $\sigma_i(G)$ is isomorphic with G and in $\sigma_i(G)$ all components are identical excepting those that correspond to V_i. It results that the number of isomorphic graph with G generated by permutation of type σ_i, for i fixed, equals tre number of permutation of V_i and so is equal with $p_i!$.

Because we fix the component i independently, it means that for any $i = 1, 2, \ldots, k$ we have $p_i!$ possibilities.

Now we can generate a permutation τ of V for which $\tau(G)$ preserves the components $V_1, V_2, \ldots, V_k$ in this order, $\tau(G)$ is isomorphic with G and

$$\tau = \sigma_1 \sigma_2 \ldots \sigma_k.$$

Because of the independency in fixing component i, the number permutation τ is equal with

$$\prod_{i=1}^{k} p_i!.$$

We considered original order for component of G, namely $V_1, V_2, \ldots, V_k$. This constraint can not be eliminated without modify the walks matrix.

We showed the following result:

Lemma 4.1 *If $G = (V, E)$ is a graph with k components, each component with p_i vertices and for any $1 \le i, j \le k$, $i \ne j$, $p_i \ne p_j$, then the number of isomorphic graph H with G for which H has a diagonal cell type walks matrix is:*

$$\prod_{i=1}^{k} p_i!.$$

To complete this result we must observe that if G is a connected graph, then the number of components is $k = 1$ and the number of vertices is $p_1 = n$. We have $p_1! = n!$ isomorphic graph H with G and H has a diagonal cell type walks matrix with $D_G = D_H$ and so the result presented by Lemma 2.1 remains valid.

The results presented in sections 3 and 4 represent solutions for two particular cases for the given problem. To solve completely the problem we must consider the general case of a graph G and to demonstrate that what we obtain remains valid in particular cases from above.

5. **General case**

Let $G = (V, E)$ be a graph with n vertices. Without losing generality we can consider that G is disconnected. If G is connected, from the result presented at the end of section 2 we have $n!$ isomorphic graph H with G and H has a diagonal cell type walks matrix with $D_G = D_H$, which is the solution to our problem.

For G disconnected there exists a partition $V_1, V_2, \ldots V_k$ of V so that if

$$E_i = \{\{x, y\} \in V | x, y \in V_i\}$$

and $G_i = (V_i, E_i)$, then $G_1, G_2, \ldots, G_k$ are components of G.

We consider that G has a diagonal cell type walks matrix D_G.

Let $\{p_1, p_2, \ldots, p_r\}$ be the set of numbers of vertices in $V_1, V_2, \ldots, V_k$, where $1 \leq r \leq k$. If $r = 1$ then all sets $V_1, V_2, \ldots, V_k$ have the same number of vertices and this is the case analyzed in section 3. If $r = k$ then any set V_i has p_i vertices and it follows that $p_i \neq p_j$ for $1 \leq i, j \leq k$, $i \neq j$ which is the case analyzed in section 3. So without losing generality we can assume that $1 < r < k$. In this case it results that there exists $i \neq j$, $1 \leq i, j \leq k$ so that V_i and V_j have the same number of vertices. Now let's consider that for any i, $1 \leq i \leq r$, c_i components in G have the number of vertices equal with p_i.

From the above consideration it results that

$$\sum_{i=1}^{k} c_i = k$$

and

$$\sum_{i=1}^{r} c_i p_i = n.$$

In the same way in which we work in sections 3 and 4 we can fix a component i, $1 \leq i \leq k$, and consider a permutation σ_i of V so that $\sigma_i(j) = j$ if $j \in V - V_i$. Because σ_i changes values only inside V_i it follows that the number of permutation of type σ_i is equal with the number of permutation of V_i, so is $p_i!$.

If we extend the fixing on components with p_j vertices, $1 \leq j \leq r$, namely $V_{s_1}, V_{s_2}, \ldots, V_{s_{c_j}}$ then we can define a permutation τ_{p_j} of V so that

$$\tau_{p_j} = \sigma_1 \sigma_2 \ldots \sigma_{c_j}.$$

This permutation changes values only inside the sets $V_{s_1}, V_{s_2}, \ldots, V_{s_{c_j}}$ and because $\sigma_1, \sigma_2, \ldots, \sigma_{c_j}$ are chosen independently it follows that the number of permutation of type τ_{p_j} is equal with $(p_j!)^{c_j}$.

Using the results from above we can now define a permutation ϵ of V so that $\epsilon = \tau_{p_1} \tau_{p_2} \ldots \tau_{p_r}$. Because we fix independently the components with the same number p_j of vertices it follows that the number of permutation of type ϵ is equal with

$$\prod_{j=1}^{r} (p_j!)^{c_j}.$$

To obtain this result we consider a given components order for G, namely the original order $V_1, V_2, \ldots, V_k$. If this constraint doesn't exist and we change the place of different dimension components then for isomorphic graph H we have $D_G \neq D_H$. Instead, if we change the place of components with the same dimension we have a isomorphic graph H with $D_G = D_H$.

Fixing i, $1 \leq i \leq r$, and so identifying the components with p_i vertices, namely $V_{s_1}, V_{s_2}, \ldots, V_{s_{c_i}}$, we must consider all possible order of this components in $V_1, V_2, \ldots, V_k$ and we have $c_i!$ possibilities. So, for i fixed, we have

$$c_i!(p_i!)^{c_i}$$

graph H, isomorphic with G with $D_G = D_H$.

So we showed the following result which gives the solution for our problem:

Theorem *If $G = (V, E)$ is a graph with k component so that the set of number of vertices is $\{p_1, p_2, \ldots, p_r\}$ and the number of components with p_i vertices is c_i for any $i = 1, 2, \ldots, r$, then the number of graph H isomorphic with G and H has a diagonal cell type walks matrix is*

$$\prod_{j=1}^{r} c_j!(p_j!)^{c_j}.$$

6. **Conclusion and remarks**

The subject of this paper is not a new one. What I've try to do is to offer a possible solving using only combinatorics tools.

Proposed solution could indicate a practical way to generate all isomorphic graphs with a given one with diagonal cell type matrix which has the same walks matrix.

To continue this subject I wish to study the corresponding problem for digraphs using paths matrix.

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Finally I want to remark that all preliminary definitions and results are the English version for those in [2].

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