ON THE EXPONENTIAL DIOPHANTINE EQUATION $2^x + 1009^y = p^z$

VIRGOLICI, Horia
Faculty of Mathematics and Computer Science
Spiru Haret University, Romania
hvirgolici@yahoo.com

Abstract
The aim of this paper is to give all nonnegative solutions (x, y, z) to the equation $2^x + 1009^y = p^z$*, where p is a positive rational prime number with* $3 \leq p \leq 997$ *(we discuss 167 equations).*

Keywords: *exponential Diophantine equation, Lebesgue-Nagell equation, Catalan equation*

AMS Classification: 11D61, 11Y50

1. **Introduction**

It is known that the equation

$$a^x + b^y = c^z \tag{1}$$

where a, b, c are prime numbers, has only finitely many solutions, but there is no algorithm to compute all the solutions (x, y, z). Some particular cases were treated: Nagell [8] found all solutions for $\max(a, b, c) = 7$, Makowski [5], Hadano [4], Uchiyama [10], Qi Sun and Xiaoming Zhou [9], and Xiaozhuo Yang [11] determined all solutions for $11 \leq \max(a, b, c) \leq 23$. Cao [3] found all solutions for $29 \leq \max(a, b, c) \leq 97$ (60 solutions in total).

The aim of this paper is to give all nonnegative solutions to equation (1) for $a = 2, b = 1009$ (1009 representing the first prime number > 1000) and c is a rational prime number, $3 \leq c \leq 997$. The main results are given by the next theorem.

Theorem 1. *The only equations* $2^x + 1009^y = p^z$*, with p rational prime,* $3 \leq p \leq 1009$*, which admit nonnegative solutions (x,y,z) are (taking* $a < b$*):*

- $2^x + 1009^y = 3^z$*, which has the solutions (1,0,1) and (3,0,2).*
- $2^x + 1009^y = 5^z$*, which has the solution (2,0,1).*
- $2^x + 1009^y = 17^z$*, which has the solution (4,0,1).*
- $2^x + 1009^y = 257^z$*, which has the solution (8,0,1).*

The proof for Theorem 1 is given in subsection 3.

2. **Preliminaries**

Below we present some theorems which establish the maximum number of solutions for the equation (1). Theorem 2 shows that the Catalan's equation has only one solution and gives this solution.

Theorem 2. ([1, 6, 7]). *Equation (named Catalan's equation)*

$$a^x - b^y = 1 \tag{2}$$

has no solutions in integers a, b, x, $y > 1$ other than $3^2 - 2^3 = 1$.

The theorem presented below is concerned with some particular cases of the equation (1).

Y. Bugeaud, M. Mignotte, S. Siksek solved completely the Lebesgue-Nagell equation

$$x^2 + D = y^n \tag{3}$$

where x, y are integers, n ≥ 3 and $1 \leq D \leq 100$. From that paper, we present only the case D = 16 (which interests us) in the following

Theorem 3. ([2]) *If D=16, equation (3) has solutions (x, y, n)=(0, 2, 4), (4, 2, 5). No other solutions exist in this case.*

3. **Proofs of the main results**

We give the proof for Theorem 1, which treats the equation

$$2^x + 1009^y = p^z,\ 3 \leq p \leq 997, p \text{ rational prime} \tag{4}$$

Many of equations (4) have no solutions (x, y, z) due to:

Lemma 1. *If $p \equiv$ 1,7,11,13,19,23,25,29,31,35,37,41,43,47 (mod 48) then the equation (4) has no nonnegative solutions (x,y,z).*

Proof: It is obvious, taking the equation (4) mod 48.

Thus, we discuss the following equations, in which p doesn't comply with the conditions in Lemma 1:

i/ $2^x + 1009^y = p^z$, where $p \in \{53, 197, 293\}$; by taking the equation mod 24, it results that $x = 2$; by taking the equation mod 7, it results that the equation has no solutions.

ii/ $2^x + 1009^y = p^z$, where $p \in \{101, 149\}$; by taking the equation mod 24, it results that $x = 2$; by taking the equation mod 5, it results that the equation has no solutions.

iii/ $2^x + 1009^y = p^z$, where $p \in \{389, 401, 449, 641, 821, 881, 929\}$; by taking the equation mod 15, it results that the equation has no solutions.

iv/ $2^x + 1009^y = 3^z$; if y=0 the equation has solutions (x,y,z)=(1,0,1) (obviously) and (3,0,2), due to Theorem 2; we consider next y $\neq 0$; by taking the equation mod 3, it results that x is odd; by taking the equation mod 16, it results one of three possibilities:

(a) x=1, which leads to equation $2+1009^y = 3^z$; by taking the equation mod 9, it results that the equation has no solutions.

(b) x=3, which leads to the equation $8 + 1009^y = 3^z$; by taking the equation mod 16, it results $z \equiv 2 (\text{mod } 4)$; by taking the equation mod 5, it results that y is even; by taking the equation mod 13, it follows that $y \equiv 0 (\text{mod } 4)$ and $z \equiv 2 (\text{mod } 12)$; by taking the equation mod 17, it results that $y \equiv 0 (\text{mod } 16)$ and $z \equiv 2 (\text{mod } 48)$; by taking the equation mod 97, it results that $y \equiv 0 (\text{mod } 96)$, and by taking the equation mod 81 it results that there are no solutions.

(c) z$\equiv$ 0(mod4) and x $>$ 4; by taking the equation mod 5, it results that x$\equiv$ 1(mod 4) and y is odd; by taking the equation mod 7, it results that x$\equiv$ 0(mod 3) and z$\equiv$ 2(mod 6); then it results that x$\equiv$ 9(mod 12) and z$\equiv$ 8(mod12); by taking now the equation mod 1009, it results that the equation has no solutions.

In conclusion, the equation iv/ has only the solutions (1,0,1) and (3,0,2).

v/ $2^x + 1009^y = 5^z$; by taking the equation mod 24, it results that x=2 and z is odd, which leads to the equation $4 + 1009^y = 5^z$; by taking this equation mod 13, it results that $y \equiv 0 (\text{mod } 4)$ and $z \equiv 1 (\text{mod } 4)$; if z=1, the equation has solution (2,0,1); if z$\geq$ 3, by taking the equation mod 7, it results that $z \equiv 1 (\text{mod } 12)$; by taking the equation mod 17, it results that $y \equiv 0 (\text{mod } 8)$; by taking the equation mod 25, it results that $y \equiv 8 (\text{mod } 40)$; by taking the equation mod 1009, it results that there are no solutions.

vi/ $2^x + 1009^y = 17^z$; by taking the equation mod 3, it results that x is even and z is odd; by taking the equation mod 16, it results that x$\geq$ 4; by taking the equation mod 32 it results one of the possibilities:

(a) y is odd and x$>$ 4; by taking the equation mod 17, it results that the equation has no solutions.

(b) x=4 and y is even; for z=1 the equation has solution (4,0,1) and for z$\geq$ 3, the equation has no solutions, due to Theorem 3.

vii/ $2^x + 1009^y = 113^z$; by taking the equation mod 7, it results that the equation has no solutions.

viii/ $2^x + 1009^y = 257^z$; by taking the equation mod 3, it results that x is even and z is odd; by taking the equation mod 16, it results that x$\geq$ 4; by taking the equation mod 32, it results one of the possibilities:

(a) x=4 and y is odd, so the equation becomes $16 + 1009^y = 257^z$; by taking this equation mod 5, it results that the equation has no solutions.

(b) x $\geq$ 6 and y is even; if y=0 the equation has the solution (8,0,1) and it is the only solution in this case due to Theorem 2; if y $\geq$ 2, by taking the equation mod 252, it results that $x \equiv 2 (\text{mod } 6)$

and $z \equiv 1 (\text{mod } 6)$; by taking the equation mod 13, it results that $x \equiv 8 (\text{mod } 12)$ and $y \equiv 0 (\text{mod } 4)$; by taking the equation mod 1009, it results that there are no solutions.

ix/ $2^x + 1009^y = 353^z$; by taking the equation mod 3, it results that x is even and z is odd; by taking the equation mod 16, it results that x $\geq$ 4; by taking the equation mod 32, it results one of the possibilities:

(a) x=4 and y is odd, so the equation becomes $16 + 1009^y = 353^z$; by taking this equation mod 5, it results that the equation has no solutions.

(b) x $\geq$ 6 and y is even; if y=0 the equation has no solutions; if y $\geq$ 2, by taking the equation mod 5, it results that $x \equiv 0 (\text{mod } 4)$ and $z \equiv 3 (\text{mod } 4)$; by taking the equation mod 17, it results that there are no solutions.

x/ $2^x + 1009^y = 593^z$; by taking the equation mod 24, it results that x is even and at least 4 and z is odd; by taking the equation mod 48, it results that y is even; if y=0, the equation has no solutions; if y $\geq$ 2, by taking the equation mod 5, it results that $x \equiv 0 (\text{mod } 4)$ and $z \equiv 3 (\text{mod } 4)$; by taking the equation mod 1009, it results that there are no solutions.

xi/ $2^x + 1009^y = 677^z$; by taking the equation mod 24, it results that x=2, so the equation becomes $4 + 1009^y = 677^z$, with z odd; by taking this equation mod 65, it results that the equation has no solutions.

xii/ $2^x + 1009^y = 773^z$; by taking the equation mod 24, it results that x=2, so the equation becomes $4 + 1009^y = 773^z$, with z odd; by taking this equation mod 9, it results that the equation has no solutions.

xiii/ $2^x + 1009^y = 977^z$; by taking the equation mod 3, it results that x is even and z is odd; by taking the equation mod 16, it results that x $\geq$ 4; by taking the equation mod 32, it results one of the possibilities:

(a) x=4 and y is even, so the equation becomes $16 + 1009^y = 977^z$, which has no solutions due to Theorem 3.

(b) x $\geq$ 6 and y is odd; by taking the equation mod 7, it results that x $\equiv$ 0 (mod 3); by taking the equation mod 5, it results that x $\equiv$ 2 (mod 4) and y $\equiv$ 1 (mod 4); from these two relations, it results that x $\equiv$ 6 (mod 12); by taking the equation mod 61 and taking into account that x $\equiv$ 6 (mod 12) and y $\equiv$ 1 (mod 4), it results that the equation has no solutions.

References

1. Bilu, Yu.F., *Catalan's Conjecture (after Mihăilescu)*, "Séminaire Bourbaki", Exposé 909, 55ème année, 2002-2003.
2. Bugeaud, Y., Mignotte, M., Siksek, S., *Classical and Modular Approaches to Exponential Diophantine Equations. II. The LebesgueNagell Equation*, "Compos. Math.", 142, 31-62, 2006.
3. Cao, Z. F., *On the Diophantine Equation* $a^x = b^y + c^z$, I , "Chinese Sci. Bull.", 32, 1519-1521, 1987; II, ibid., 33, 237, 1988 (in Chinese).
4. Hadano, T., *On the Diophantine Equation* $a^x = b^y + c^z$, "Math. J. Okayama Univ.", 19, 25-29, 1976/77.
5. Makowski, A., *On the Diophantine Equation* $2^x + 11^y = 5^z$, "Nord. Mat. Tidskr.", 7, 81-96, 1959.
6. Mihailescu, P., *A Class Number Free Criterion for Catalan's Conjecture*, "J. of Number Theory", 99, 225-231, 2003.
7. Mihailescu, P., *Primary Cyclotomic Units and a Proof of Catalan's Conjecture*, "Journal für die reine und angewandte Mathematik", 572, 167-195, 2004.
8. Nagell, T., *Sur une classe d'équations exponentielles*, "Ark. Mat.", 3, 569-582, 1958.
9. Sun, Q., Zhou, X. M., *On the Diophantine Equation* $a^x = b^y + c^z$, "Chinese Sci. Bull.", 29, 61, 1984 (in Chinese).
10. Uchiyama, S., *On the Diophantine Equation* $2^x = 3^y + 13^z$, "Math. J. Okayama Univ.", 19, 31-38, 1976/77.
11. Yang, X. Z., *On the Diophantine Equation* $a^x = b^y + c^z$, "Sichuan Daxue Xuebao", 4, 151-158, 1985 (in Chinese).