ON COMPACT EINSTEIN WARPED PRODUCTS

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Abstract

The aim of this paper is to give some obstructions to the existence of compact Einstein warped products. It is proven in ([7]) that a compact Einstein warped product should have strictly positive scalar curvature. Also, in ([8]) it is proven that the dimension of the base of a compact Einstein warped product cannot be two. In this paper we prove that the fibre of a compact Einstein warped product should have strictly positive scalar curvature and we give some new obstructions to the existence of such spaces. In particular, we remark that a compact Einstein warped product must have dimension greater than five.

Keywords: *warped products, Einstein spaces, warping function*

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1. Introduction

The notion of warped product manifold was introduced in ([2]) where it served to give new examples of Riemannian manifolds. This kind of methods are used to construct Einstein metrics on noncompact complete manifolds or other examples in differential geometry. In ([1]), A.L. Besse asks the following question: "Does there exist a compact Einstein warped product with non-constant warping function?". Clearly a Riemannian product $M = N \times F$ is Einstein if N and F are Einstein manifolds with the same scalar curvature. In the case of a non-trivial warped product the answer to Besse's question is more difficult.

Let $(N, g_N), (F, g_F)$ be two Riemannian manifolds, $\dim N = m \geq 1$, $\dim F = k \geq 1$ and $f : N \to (0, \infty)$, $f \in C^\infty(N)$. The warped product, $M = N \times_f F$, is the Riemannian manifold $N \times F$ furnished with the metric $g_M = g_N + f^2 g_F$. The manifold N is called the base of M, the manifold F is called the fibre of M and the warped product is called trivial if the warping function f is a constant. We will denote by Ric_M, Ric_N, Ric_F and H^f the Ricci curvature of M, the lifts to M of the Ricci curvatures of N and F and the Hessian of f, respectively. A Riemannian manifold is said to be Einstein if its Ricci tensor is proportional to the metric, that is $Ric_M = \lambda g_M$, $\lambda \in R$.

By τ_M, τ_N and τ_F we will understand the scalar curvatures of M, N and F, that is $\tau_M = Tr(Ric_M)$, $\tau_N = Tr(Ric_N)$ and $\tau_F = Tr(Ric_F)$.

2. **Compact Einstein warped products**

We present the Ricci tensor of a warped product.

Proposition 2.1. *([2]) The Ricci curvature Ric_M of the warped product $M = N \times_f F$ satisfies:*

a). $Ric_M(X,Y) = Ric_N(X,Y) - \frac{k}{f}H^f(X,Y)$,

b). $Ric_M(X,V) = 0$,

c). $Ric_M(V,W) = Ric_F(V,W) - g_M(V,W)[-\frac{\triangle f}{f} + \frac{k-1}{f^2} g_N(\bigtriangledown f, \bigtriangledown f)]$,

for any horizontal vectors X, Y and any vertical vectors V, W, where $\triangle f$ denotes the Laplacian of f given by $\triangle f = -Tr(H^f)$ and $\bigtriangledown f$ denotes the gradient of f.

Thus the Einstein equations become:

Corollary 2.1. *([7]) The warped product $M = N \times_f F$ is an Einstein space with $Ric_M = \lambda g_M$ if and only if the followings hold:*

(1.1) $Ric_N = \lambda g_N + \frac{k}{f} H^f$,

(1.2) *(F, g_F) is an Einstein space with $Ric_F = \mu g_F$, for a constant μ ,*

(1.3) $-f \triangle f + (k-1) | \bigtriangledown f |^2 + \lambda f^2 = \mu$.

Remark 2.1. ([1]) In the conditions of corollary 2.1., by taking the traces in the first two equations (1.1) and respectively (1.2), we obtain the followings:

(1.4) $\tau_N = m\lambda - k\frac{\triangle f}{f}$.

(1.5) $\tau_F = k\mu$.

Now, we combine the relations (1.3), (1.4) and (1.5) to obtain a formula for the scalar curvature of M.

Remark 2.2. ([1]) In the conditions of corollary 2.1., by using (1.3), (1.4) and (1.5) we obtain the following formula:

(1.6) $\tau_N + \frac{\tau_F}{f^2} = (m+k)\lambda + k(k-1)\frac{|\nabla f|^2}{f^2} - 2k\frac{\triangle f}{f} = \tau_M + k(k-1)\frac{|\nabla f|^2}{f^2} - 2k\frac{\triangle f}{f}$,

since $\tau_M = \lambda(m+k)$.

We give now some obstructions to the existence of compact Einstein warped products.

Remark 2.3. ([1]) Let $M = N \times_f F$ be a warped product with N compact, $\dim N = m \geq 1$ and $\dim F = k \geq 1$, which is also an Einstein space with $Ric_M = \lambda g_M$. If $m = 1$ or $k = 1$, then M is a simply Riemannian product.

Indeed,

i). If $m = 1$, then $\tau_N = 0$. So, from (1.4) we obtain that

$$0=\lambda-k\frac{\triangle f}{f}\Rightarrow$$
$$\frac{\triangle f}{f}=\frac{\lambda}{k}>0\text{ (or }<0)\Rightarrow$$
$$\triangle f>0\text{ (or }<0)$$

Thus, f is constant.

ii). If $k=1$, then $\tau_F=0$. Hence, from (1.5) we have $\mu=0$. Thus, (1.3) becomes:

$$-f\triangle f+\lambda f^2=0\Rightarrow$$
$$\triangle f=\lambda f>0\text{ (or }<0)$$

Hence, f is constant.

We have the following result proven in ([7]):

Theorem 2.1. *([7]) Let $M=N\times_f F$ be a warped product with N compact, $\dim N=m\geq 2$ and $\dim F=k\geq 2$, which is also an Einstein space with $Ric_M=\lambda g_M$. If $\lambda\leq 0$, then M is a simply Riemannian product.*

Thus, in the rest of the paper we will consider $\lambda>0$.

Theorem 2.2. *([8]) Let $M=N\times_f F$ be a warped product with N compact, $\dim N=m\geq 2$ and $\dim F=k\geq 2$, which is also an Einstein space with $Ric_M=\lambda g_M$. If $m=2$, then M is a simply Riemannian product.*

Theorem 2.3. *([9]) Let $M=N\times_f F$ be a warped product with N and F compact, $\dim N=m\geq 2$ and $\dim F=k\geq 2$, which is also an Einstein space with $Ric_M=\lambda g_M$. Suppose that $N=N_1\times N_2$, where the scalar curvature of N_1, τ_{N_1}, is a positive constant and the scalar curvature of N_2, τ_{N_2}, verifies the condition*

$$\int_{N_2}\tau_{N_2}dvol(g_{N_2})\leq 0,$$

with $|\tau_{N_1}|\leq\tau_{N_2}$. Then M is a simply Riemannian product.

From now on, throughout this paper we will consider that the base has the dimension greater than three and the fibre has the dimension greater than two.

Remark 2.4. We deduce that a compact Einstein warped product must have dimension greater than 5.

Theorem 2.4. *Let $M=N\times_f F$ be a warped product with N compact, $\dim N=m\geq 3$ and $\dim F=k\geq 2$, which is also an Einstein space with $Ric_M=\lambda g_M$. If the scalar curvature of F is not strictly positive, then M is a simply Riemannian product.*

Proof: We have from the hypothesis that $\tau_F\leq 0$. Thus, (1.5) implies $\mu\leq 0$.

So, from (1.3) we have that:

$$
\begin{gathered}
-f \triangle f + (k-1)|\bigtriangledown f|^2 + \lambda f^2 = \mu \Rightarrow \\
-f \triangle f + \lambda f^2 = \mu - (k-1)|\bigtriangledown f|^2 \leq 0 \Rightarrow \\
f \triangle f \geq \lambda f^2 \Rightarrow \\
\triangle f \geq \lambda f > 0.
\end{gathered}
$$

Hence, f is constant.

Thus, in the rest of the paper we will also consider $\mu > 0$.

Theorem 2.5. *Let $M = N \times_f F$ be a warped product with N compact, $\dim N = m \geq 3$ and $\dim F = k \geq 2$, which is also an Einstein space with $Ric_M = \lambda g_M$. If at least one of the following conditions a).- f). is true, then M becomes a simply Riemannian product.*

a). $|\bigtriangledown f| \geq \sqrt{\frac{\mu}{k-1}}$,
b). $\tau_M \leq \tau_N$,
c). $\tau_M \geq \tau_N + \tau_F$ and $\mu \geq \lambda$,
d). $\tau_M \geq \tau_N + \frac{\tau_F}{f^2}$,
e). $\tau_N \leq 0$,
f). $\tau_N \geq \frac{m\mu}{f^2}$ and $m \geq k$.

Proof: a). From (1.3) we have that:

$$
\begin{gathered}
-f \triangle f + (k-1)|\bigtriangledown f|^2 + \lambda f^2 = \mu \Rightarrow \\
-f \triangle f + \lambda f^2 = \mu - (k-1)|\bigtriangledown f|^2 \leq 0 \Rightarrow \\
-f \triangle f + \lambda f^2 \leq 0 \Rightarrow \\
f \triangle f \geq \lambda f^2 \Rightarrow \\
\triangle f > 0.
\end{gathered}
$$

Thus, f is constant.

b). From (1.4) we have that:

$$
\begin{gathered}
\tau_N = m\lambda - k\frac{\triangle f}{f} \Rightarrow \\
\tau_N + k\lambda = m\lambda - k\frac{\triangle f}{f} + k\lambda = \tau_M - k\frac{\triangle f}{f} \Rightarrow \\
\tau_M - \tau_N = k\lambda + k\frac{\triangle f}{f} = k[\frac{\triangle f}{f} + \lambda] \leq 0 \Rightarrow \\
\frac{\triangle f}{f} + \lambda \leq 0 \Rightarrow \\
\frac{\triangle f}{f} \leq -\lambda < 0 \Rightarrow \\
\triangle f < 0.
\end{gathered}
$$

Hence, f is constant.

c). From (1.5) we have that $\tau_F = k\mu \geq k\lambda$. Thus, together with (1.4) we get:

$$\tau_F + \tau_N \geq k\lambda + m\lambda - k\frac{\triangle f}{f} = \tau_M - k\frac{\triangle f}{f} \Rightarrow$$

$$k\frac{\triangle f}{f} \geq \tau_M - (\tau_F + \tau_N) \geq 0 \Rightarrow$$
$$\triangle f \geq 0.$$

Hence, f is constant.

d). From (1.6) we have that:

$$\tau_N + \frac{\tau_F}{f^2} = \tau_M + k(k-1)\frac{|\nabla f|^2}{f^2} - 2k\frac{\triangle f}{f} \Rightarrow$$
$$\tau_M - (\tau_N + \frac{\tau_F}{f^2}) = 2k\frac{\triangle f}{f} - k(k-1)\frac{|\nabla f|^2}{f^2} \geq 0 \Rightarrow$$
$$2k\frac{\triangle f}{f} \geq k(k-1)\frac{|\nabla f|^2}{f^2} \geq 0 \Rightarrow$$
$$\triangle f \geq 0.$$

Thus, f is constant.

e). From (1.4) we have that:

$$\tau_N = m\lambda - k\frac{\triangle f}{f} \Rightarrow$$
$$k\frac{\triangle f}{f} = m\lambda - \tau_N \geq 0 \Rightarrow$$
$$\triangle f \geq 0.$$

Hence, f is constant.

f). From (1.4) we have that:

$$\tau_N = m\lambda - k\frac{\triangle f}{f} \Rightarrow$$
$$\tau_N f^2 = m\lambda f^2 - kf \triangle f$$

Also, from (1.3) we have that:

$$-f \triangle f + (k-1)|\bigtriangledown f|^2 + \lambda f^2 = \mu \Rightarrow$$
$$\lambda f^2 = \mu + f \triangle f - (k-1)|\bigtriangledown f|^2$$

Hence

$$\tau_N f^2 = m[\mu + f \triangle f - (k-1)|\bigtriangledown f|^2] - kf \triangle f =$$
$$(m-k)f \triangle f + m\mu - m(k-1)|\bigtriangledown f|^2 \Rightarrow$$
$$\tau_N f^2 - m\mu = (m-k)f \triangle f - m(k-1)|\bigtriangledown f|^2 \geq 0.$$

We suppose now that $m \geq k$. We consider two situations:
i). $m = k$ implies:

$$-m(m-1)|\bigtriangledown f|^2 \geq 0 \Rightarrow$$
$$m(m-1)|\bigtriangledown f|^2 \leq 0 \Rightarrow |\bigtriangledown f|^2 = 0 \Rightarrow \bigtriangledown f = 0$$

Hence, f constant.

ii). $m > k$ implies:

$$(m-k)f \triangle f \geq m(k-1)|\bigtriangledown f|^2 \geq 0 \Rightarrow$$
$$\triangle f \geq 0.$$

Thus, f is constant.

References

1. Besse, A.L., *Einstein Manifolds*, Springer-Verlag, 1987.
2. Bishop, R.L., O'neill, B., *Manifolds of negative curvature*, Trans. Amer. Math. Soc., 145, 1-49, 1969.
3. Dobarro, F., Dozo, L. E., *Scalar curvatures and warped products of Riemannian geometry*, Trans. Am. Math. Soc., 303, 161-168, 1987.
4. Dumitru, D., *On Einstein spaces of odd dimension*, Bulletin of the Transilvania University of Braşov, Supplement, 14, 49, Series B, 95-97, 2007.
5. Dumitru, D., *On multiple generalized Robertson-Walker space-time*, accepted for publication in Annals of the University of Oradea, Mathematical Series, 2011.
6. Easley, K.L., *Local existence of warped product metrics*, PhD. thesis, University of Missouri-Columbia, 1991.
7. Kim, D.S., Kim, Y.H., *Compact Einstein warped product spaces with nonpositive scalar curvature*, Proc. Am. Math. Soc, 131, 2573-2576, 2003.
8. Kim, D.S., *Compact Einstein warped product spaces*, Trends in Mathematics, Information center for mathematical sciences, 5, 2, 1-5, 2002.
9. Kim, S., *Warped product and Einstein metrics*, J. Phys. A: Math. Gen., 39, L329-L333, 2006.
10. Mustafa, M.T., *A non-existence result for compact Einstein warped products*, J. Phys. A: Math. Gen., 38, L791-L793, 2005.
11. Palais, R.S., Derdzinski, A., Terng, C.L., *Warped products and Einstein Manifolds*, Proceedings of Differential Geometry Meeting, Münster, 44-47, 1982.