

APPROXIMATION OF FUNCTIONS OF TWO VARIABLES BY DISCRETE LINEAR POSITIVE OPERATORS

VORNICESCU, Neculae

Faculty of Finance and Banking, Blaj

Spiru Haret University

nvornicescu@yahoo.com

Abstract

We introduce a class of discrete linear positive operators in the space of real continuous functions of two variables defined on the square $[0, 1] \times [0, 1]$. These operators are associated to an arbitrary set of nodes (x_i, y_j) , $i = 0, \dots, m$, $j = 0, \dots, n$, and preserve the linear functions.

Keywords: linear positive operators, polynomials

AMS Classification: 41A35, 41A36

1. Introduction

The most results concerning the approximation of continuous functions on the compact domain $D = [0, 1] \times [0, 1]$ by polynomials use some linear positive operators which preserve the linear functions. Some discrete operators are of the form

$$T_{m,n}(f)(x, y) = \sum_{i=0}^m \sum_{j=0}^n f(x_i, y_j) u_i(x) v_j(y), \quad (1)$$

where p_i are polynomials of degree m and q_j are polynomials of degree n .

An example is Bernstein operator

$$B_{m,n}(f)(x, y) = \sum_{i=0}^m \sum_{j=0}^n f\left(\frac{i}{m}, \frac{j}{n}\right) \binom{m}{i} x^i (1-x)^{m-i} \binom{n}{j} y^j (1-y)^{n-j}.$$

This operator has the well known properties:

$$B_{m,n}(e_0)(x, y) = 1,$$

$$B_{m,n}(e_1)(x, y) = x,$$

$$B_{m,n}(h_1)(x, y) = y,$$

$$B_{m,n}(e_2)(x, y) = x^2 + \frac{1}{m} x(1-x),$$

$$B_{m,n}(h_2)(x, y) = y^2 + \frac{1}{n} y(1-y),$$

where $e_0(x, y) = 1$, $e_1(x, y) = x$, $e_2(x, y) = x^2$, $h_1(x, y) = y$, and $h_2(x, y) = y^2$.

The aim of this paper is to give a characterization of those operators (1) for which there exist the real constants a and b such that

$$T_{m,n}(e_0)(x, y) = 1, \quad (2)$$

$$T_{m,n}(e_1)(x, y) = x, \quad (3)$$

$$T_{m,n}(h_1)(x, y) = y, \quad (4)$$

$$T_{m,n}(e_2)(x, y) = x^2 + ax(1 - x), \quad (5)$$

$$T_{m,n}(h_2)(x, y) = y^2 + by(1 - y). \quad (6)$$

2. Main results

Let $\Delta_1 = \{x_0 = 0 < x_1 < x_2 < \dots < x_m = 1\}$, $\Delta_2 = \{y_0 = 0 < y_1 < y_2 < \dots < y_n = 1\}$ and $\Delta = \Delta_1 \times \Delta_2$ be the fixed sets of nodes. For two sets of polynomials $P = \{p_1, p_2, \dots, p_{m-1}\}$ and $Q = \{q_1, q_2, \dots, q_{n-1}\}$ satisfying

$$\sum_{i=1}^{m-1} p_i(x) = 1, \text{ for all } x \in [0, 1], \quad \sum_{j=1}^{n-1} q_j(y) = 1, \text{ for all } y \in [0, 1]$$

and for three real numbers α, β, A , $A \neq 0$, we will build an operator which satisfies (2)-(6). First we construct the following polynomials:

$$u_0(x) = A(1 - x) \left[1 - (1 - \alpha)x \sum_{k=1}^{m-1} \frac{1}{x_k} p_k(x) \right], \quad (7)$$

$$u_i(x) = A \frac{1 - \alpha}{x_i(1 - x_i)} x(1 - x) p_i(x), \quad i = 1, 2, \dots, m - 1,$$

$$u_m(x) = Ax \left[1 - (1 - \alpha)(1 - x) \sum_{k=1}^{m-1} \frac{1}{1 - x_k} p_k(x) \right],$$

$$v_0(y) = \frac{1}{A} (1 - y) \left[1 - (1 - \beta)y \sum_{k=1}^{n-1} \frac{1}{y_k} q_k(y) \right],$$

$$v_j(y) = \frac{1}{A} \frac{1 - \beta}{y_j(1 - y_j)} y(1 - y) q_j(y), \quad j = 1, 2, \dots, n - 1,$$

$$v_n(y) = \frac{1}{A} y \left[1 - (1 - \beta)(1 - y) \sum_{k=1}^{n-1} \frac{1}{1 - y_k} q_k(y) \right].$$

Then we consider the operator $T_{P,Q,\alpha,\beta,A}$ defined on $C([0, 1] \times [0, 1])$, with values in the set of polynomials in two variables

$$T_{P,Q,\alpha,\beta,A}(f)(x, y) = \sum_{i=0}^m \sum_{j=0}^n f(x_i, y_j) u_i(x) v_j(y) \quad (8)$$

Lemma 1. *The operator $T_{P,Q,\alpha,\beta,A}$ defined by (8) has the properties*

$$\begin{aligned} T(e_0)(x, y) &= 1, \\ T(e_1)(x, y) &= x, \\ T(h_1)(x, y) &= y, \\ T(e_2)(x, y) &= x^2 + \alpha x(1 - x), \\ T(h_2)(x, y) &= y^2 + \beta y(1 - y). \end{aligned}$$

Proof: We have

$$\begin{aligned} u_i(x) &= A(1 - \alpha)x(1 - x)p_i(x) \left(\frac{1}{x_i} + \frac{1}{1 - x_i} \right), \\ x_i u_i(x) &= A \frac{(1 - \alpha)x(1 - x)}{1 - x_i} p_i(x), \\ x_i^2 u_i(x) &= A(1 - \alpha)x(1 - x) \left(\frac{1}{1 - x_i} - 1 \right) p_i(x), \quad i = 1, 2, \dots, m - 1 \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{m-1} u_i(x) &= A - u_0(x) - u_m(x), \\ \sum_{i=1}^{m-1} x_i u_i(x) &= Ax - u_m(x), \\ \sum_{i=1}^{m-1} x_i^2 u_i(x) &= Ax - u_m(x) - A(1 - \alpha)x(1 - x) = Ax^2 + A\alpha x(1 - x) - u_m(x). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{i=0}^m u_i(x) &= A \\ \sum_{i=1}^m x_i u_i(x) &= Ax \\ \sum_{i=1}^{m-1} x_i^2 u_i(x) &= A[x^2 + \alpha x(1 - x)] \end{aligned}$$

Similarly we obtain that

$$\begin{aligned}\sum_{j=0}^n v_j(y) &= \frac{1}{A}, \\ \sum_{j=1}^n y_j v_j(y) &= \frac{1}{A}y, \\ \sum_{j=1}^n y_j^2 v_j(y) &= \frac{1}{A}[y^2 + \beta y(1-y)].\end{aligned}$$

A simple calculation show that the conclusion of lemma is true.

In what follows we will consider only positive operators. Let $\mathcal{P}_{m,n}$, $m, n \geq 2$, denote the set of polynomials in two variables, of degree m in x and degree n in y and let \mathcal{P}_s^k denote the linear space

$$\mathcal{P}_s^k = \{(p_1, p_2, \dots, p_k), \text{ with } p_1, p_2, \dots, p_k \text{ polynomials of degree } s\}.$$

An element $(p_1, p_2, \dots, p_k) \in \mathcal{P}_s^k$ is said to be *admissible* if

$$p_i(x) \geq 0, \quad i = 1, 2, \dots, k \quad \text{and} \quad \sum_{i=1}^k p_i(x) = 1 \text{ for all } x \in [0, 1].$$

For an admissible element $\mathbf{p} = (p_1, p_2, \dots, p_{m-1}) \in \mathcal{P}_{m-2}^{m-1}$ we will define $\alpha(\mathbf{p})$ as

$$\alpha(\mathbf{p}) = \max \left\{ 1 - \frac{1}{m_1}, 1 - \frac{1}{m_2} \right\},$$

where

$$\begin{aligned}m_1 &= \max \left\{ x \sum_{i=1}^{m-1} \frac{1}{x_i} p_i(x), \quad x \in [0, 1] \right\}, \\ m_2 &= \max \left\{ (1-x) \sum_{i=1}^{m-1} \frac{1}{1-x_i} p_i(x), \quad x \in [0, 1] \right\}.\end{aligned}$$

We remark that

$$m_1 \geq \sum_{i=1}^{m-1} \frac{1}{x_i} p_i(1) > \sum_{i=1}^{m-1} p_i(1) = 1.$$

and $m_2 > 1$.

Similarly, for an admissible element $\mathbf{q} = (q_1, q_2, \dots, q_{n-1}) \in \mathcal{P}_{n-2}^{n-1}$, we will define $\beta(\mathbf{q})$ as

$$\beta(\mathbf{q}) = \max \left\{ 1 - \frac{1}{n_1}, 1 - \frac{1}{n_2} \right\},$$

where

$$n_1 = \max \left\{ y \sum_{j=1}^{n-1} \frac{1}{y_j} q_j(y), y \in [0, 1] \right\},$$

$$n_2 = \max \left\{ (1-y) \sum_{j=1}^{n-1} \frac{1}{1-y_j} q_j(y), y \in [0, 1] \right\}.$$

Theorem 1. *Let us consider the admissible elements \mathbf{p} and \mathbf{q} , where*

$$\mathbf{p} = (p_1, p_2, \dots, p_{m-1}) \in \mathcal{P}_{m-2}^{m-1}$$

and

$$\mathbf{q} = (q_1, q_2, \dots, q_{n-1}) \in \mathcal{P}_{n-2}^{n-1}.$$

Then the polynomials

$$u_0(x) = A(1-x)[1 - (1 - \alpha(\mathbf{p}))x \sum_{i=1}^{m-1} \frac{1}{x_i} p_i(x)],$$

$$u_i(x) = A \frac{1 - \alpha(\mathbf{p})}{x_i(1-x_i)} x(1-x)p_i(x), \quad i = 1, 2, \dots, m-1,$$

$$u_m(x) = Ax[1 - (1 - \alpha(\mathbf{p}))(1-x) \sum_{i=1}^{m-1} \frac{1}{1-x_i} p_i(x)],$$

$$v_0(y) = \frac{1}{A}(1-y)[1 - (1 - \beta(\mathbf{q}))y \sum_{j=1}^{n-1} \frac{1}{y_j} q_j(y)],$$

$$v_j(y) = \frac{1}{A} \frac{1 - \beta(\mathbf{q})}{y_j(1-y_j)} y(1-y)q_j(y), \quad j = 1, 2, \dots, n-1,$$

$$v_n(y) = \frac{1}{A} y[1 - (1 - \beta(\mathbf{q}))(1-y) \sum_{j=1}^{n-1} \frac{1}{1-y_j} q_j(y)],$$

where A is an arbitrary positive constant, are nonnegative on $[0, 1]$. Moreover, the operator $T : C([0, 1] \times [0, 1]) \rightarrow \mathcal{P}_{m,n}$,

$$T(f)(x, y) = \sum_{i=0}^m \sum_{j=0}^n f(x_i, y_j) u_i(x) v_j(y),$$

is positive and has the properties

$$T(e_0)(x, y) = 1,$$

$$T(e_1)(x, y) = x, \quad T(h_1)(x, y) = y,$$

$$T(e_2)(x, y) = x^2 + \alpha(\mathbf{p})x(1-x), \quad T(h_2)(x, y) = y^2 + \beta(\mathbf{q})y(1-y).$$

Proof: From Lemma 1 we conclude that it is enough to prove that T is positive. Obviously

$$\begin{aligned} u_i(x) &\geq 0, \text{ for } x \in [0, 1], \quad i = 1, 2, \dots, m-1, \text{ and} \\ v_j(y) &\geq 0, \text{ for } y \in [0, 1], \quad j = 1, 2, \dots, n-1. \end{aligned}$$

The inequality $u_0(x) \geq 0$ for $x \in]0, 1]$ is equivalent to

$$\alpha(\mathbf{p}) \geq 1 - \frac{1}{x \sum_{i=1}^{m-1} \frac{1}{x_i} p_i(x)},$$

which is true from the definition of $\alpha(\mathbf{p})$. Inequality $u_m(x) \geq 0$ for $x \in [0, 1[$ is equivalent to

$$\alpha(\mathbf{p}) \geq 1 - \frac{1}{(1-x) \sum_{i=1}^{m-1} \frac{1}{1-x_i} p_i(x)},$$

which is also true from the definition of $\alpha(\mathbf{p})$. Similarly for v_0 and v_n .

In the case of theorem we will said that the operator T is *generated* by \mathbf{p} and \mathbf{q} .

In the next theorem we show that each linear positive operator $T : C([0, 1] \times [0, 1]) \rightarrow \mathcal{P}_{m,n}$, having the form (1) and verifying properties (2)-(6), is generated by an admissible element $\mathbf{p} \in \mathcal{P}_{m-2}^{m-1}$ and an admissible element $\mathbf{q} \in \mathcal{P}_{n-2}^{n-1}$.

Theorem 2. *Let $T : C([0, 1] \times [0, 1]) \rightarrow \mathcal{P}_{m,n}$,*

$$T(f)(x, y) = \sum_{i=0}^m \sum_{j=0}^n f(x_i, y_j) u_i(x) v_j(y)$$

be a linear positive operator having the properties

$$\begin{aligned} T(e_0)(x, y) &= 1, \\ T(e_1)(x, y) &= x, \quad T(h_1)(x, y) = y, \\ T(e_2)(x, y) &= x^2 + ax(1-x), \quad T(h_2)(x, y) = y^2 + by(1-y). \end{aligned}$$

Then there exist the admissible elements $\mathbf{p} = (p_1, p_2, \dots, p_{m-1}) \in \mathcal{P}_{m-2}^{m-1}$, $\mathbf{q} = (q_1, q_2, \dots, q_{n-1}) \in \mathcal{P}_{n-2}^{n-1}$ and a constant $A > 0$, such that $\alpha(\mathbf{p}) \leq a$,

$\beta(\mathbf{q}) \leq b$ and

$$\begin{aligned}
u_0(x) &= A(1-x) \left[1 - (1-a)x \sum_{i=1}^{m-1} \frac{1}{x_i} p_i(x) \right], \\
u_i(x) &= A \frac{1-a}{x_i(1-x_i)} x(1-x) p_i(x), \quad i = 1, 2, \dots, m-1, \\
u_m(x) &= Ax \left[1 - (1-a)(1-x) \sum_{i=1}^{m-1} \frac{1}{1-x_i} p_i(x) \right], \\
v_0(y) &= \frac{1}{A} (1-y) \left[1 - (1-b)y \sum_{j=1}^{n-1} \frac{1}{y_j} q_j(y) \right], \\
v_j(y) &= \frac{1}{A} \frac{1-b}{y_j(1-y_j)} y(1-y) q_j(y), \quad j = 1, 2, \dots, n-1, \\
v_n(y) &= \frac{1}{A} y \left[1 - (1-b)(1-y) \sum_{j=1}^{n-1} \frac{1}{1-y_j} q_j(y) \right].
\end{aligned}$$

Proof: From $T(e_0)(x, y) = 1$ we deduce that

$$\sum_{i=0}^m u_i(x) \sum_{j=0}^n v_j(y) = 1,$$

whence $\sum_{i=0}^m u_i(x)$ and $\sum_{j=0}^n v_j(y)$ are constant. Let

$$A = \sum_{i=0}^m u_i(x). \tag{9}$$

Then,

$$\sum_{j=0}^n v_j(y) = \frac{1}{A}. \tag{10}$$

From $T(e_1)(x, y) = x$ we obtain

$$\sum_{i=1}^m x_i u_i(x) = Ax, \tag{11}$$

and from the positivity of the polynomials u_i , by taking $x = 0$, we deduce that

$$u_i(x) = x u_i^{[1]}(x), \quad i = 1, 2, \dots, n.$$

and equality (11) becomes

$$\sum_{i=1}^m x_i u_i^{[1]}(x) = A. \tag{12}$$

Further, from $T(e_2)(x, y) = x^2 + ax(1 - x)$ we obtain

$$\sum_{i=1}^m x_i^2 u_i^{[1]}(x) = A[x + a(1 - x)]. \quad (13)$$

and by subtracting (13) from (12) one gets

$$\sum_{i=1}^{m-1} x_i(1 - x_i)u_i^{[1]}(x) = A(1 - a)(1 - x). \quad (14)$$

We remark that equality (14) implies $a < 1$.

For $x = 1$ we have

$$u_i^{[1]}(x) = (1 - x)u_i^{[2]}(x), \quad i = 1, 2, \dots, m - 1.$$

Then

$$u_i(x) = x(1 - x)u_i^{[2]}(x), \quad i = 1, 2, \dots, m - 1 \quad (15)$$

and (14) becomes

$$\sum_{i=1}^{m-1} x_i(1 - x_i)u_i^{[2]}(x) = A(1 - a). \quad (16)$$

For $i = 1, 2, \dots, m - 1$ we define

$$p_i(x) = \frac{x_i(1 - x_i)}{A(1 - a)}u_i^{[2]}(x) = \frac{x_i(1 - x_i)}{A(1 - a)} \frac{u_i(x)}{x(1 - x)}. \quad (17)$$

Then p_i are positive polynomials of degree $m - 1$, and from (16) we obtain

$$\sum_{i=1}^{m-1} p_i(x) = 1.$$

From (17) we obtain

$$u_i(x) = A(1 - a)x(1 - x) \frac{p_i(x)}{x_i(1 - x_i)}, \quad i = 1, 2, \dots, m - 1.$$

Further, from (11) we deduce that

$$u_m(x) = A \left[1 - (1 - a)(1 - x) \sum_{i=1}^{m-1} \frac{p_i(x)}{1 - x_i} \right]$$

and from (9) we obtain

$$u_0(x) = A \left[1 - (1 - a)x \sum_{i=1}^{m-1} \frac{p_i(x)}{x_i} \right].$$

But $u_0(x) \geq 0$, therefore

$$x \sum_{i=1}^{m-1} \frac{p_i(x)}{x_i} \leq \frac{1}{1-a} \text{ and}$$

$$m_1 = \max \left\{ x \sum_{i=1}^{m-1} \frac{1}{x_i} p_i(x), x \in [0, 1] \right\} \leq \frac{1}{1-a}.$$

Analogous, from $u_m(x) \geq 0$ we obtain

$$m_2 = \max \left\{ (1-x) \sum_{i=1}^{m-1} \frac{1}{1-x_i} p_i(x), x \in [0, 1] \right\} \leq \frac{1}{1-a},$$

therefore $\alpha(\mathbf{p}) \leq a$.

Similarly, we can conclude that, in the case when

$$q_j(y) = \frac{Ay_j(1-y_j)}{(1-b)} \frac{v_j(y)}{y(1-y)}, \quad j = 1, 2, \dots, n-1,$$

the following equalities hold:

$$\sum_{j=1}^{n-1} q_j(y) = 1,$$

$$v_j(y) = \frac{(1-b)y(1-y)q_j(y)}{Ay_j(1-y_j)}, \quad j = 1, 2, \dots, n-1,$$

$$v_n(y) = \frac{1}{A} [1 - (1-b)(1-y) \sum_{j=1}^{n-1} \frac{q_j(y)}{1-y_j}],$$

$$v_0(y) = \frac{1}{A} [1 - (1-b)y \sum_{j=1}^{n-1} \frac{q_j(y)}{y_j}],$$

$$\beta(\mathbf{q}) \leq b.$$

Remark 1. In the hypotheses of the theorem, if $\alpha(\mathbf{p}) < a$ or $\beta(\mathbf{p}) < b$, then the operator T^* generated by \mathbf{p} and \mathbf{q} is better than T since

$$T^*(e_0)(x, y) = 1,$$

$$T^*(e_1)(x, y) = x, \quad T^*(h_1)(x, y) = y,$$

$$T^*(e_2)(x, y) = x^2 + \alpha(\mathbf{p})x(1-x), \quad T^*(h_2)(x, y) = y^2 + \beta(\mathbf{q})y(1-y).$$

References

1. Lorentz, G.G., *Bernstein polynomials*, "Univ. Toronto Press", 1967.
2. Stancu, D.D., Coman, G., Agratini, O., Trîmbişas, R., *Analiză numerică și teoria aproximării*, "Presa Universitară Clujeană", Cluj-Napoca, 2001.
3. Vornicescu, N., *A Class of Linear Positive Operators*, In A. Lupaş (ed.), "Mathematical Analysis and Approximation Theory", Burg Verlag, Sibiu, 309-315, 2002.