CONNECTEDNESS OF THE ATTRACTORS OF ITERATED FUNCTION SYSTEMS

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Abstract

We survey some of the topological properties of the attractors of iterated function systems such as connectedness, arcwise connectedness and locally arcwise connectedness. We give necessary and sufficient conditions for the attractors of finite iterated function system (IFS) to be a connected sets and in the case when infinite iterated function systems (IIFS) are considered we give only some sufficient conditions. Attractors with many connected components are also considered in the case of IFSs. We also describe the shift space associated to an attractor. Some examples are given for each considered case.

Keywords: *attractors, iterated function systems, connectednes, arcwise connectedness*

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1. Introduction

Iterated function systems were conceived in the present form by John Hutchinson in [8], popularized by Michael Barnsley in [1] and are one of the most common and general ways to generate fractals. Many of the important examples of functions and sets with special and unusual properties turn out to be fractal sets or functions whose graphs are fractal sets and a great part of them are attractors of IFSs. There is a current effort to extend the classical Hutchinson's framework to more general spaces and infinite iterated function systems (IIFSs) or, more generally, to multifunction systems and to study them ([6], [7]). Although the fractal sets are defined with measure theory, being sets with noninteger Hausdorff dimension ([2], [5]), it turns out that they have interesting topological properties ([7]).

We start with a metric space (X, d) and we denote by $\mathcal{K}(X)$ the set of nonempty compact subsets of X. For a set $A \subset X$ we denote by $d(A)$ the diameter of A, that is $d(A) = \sup\limits_{x,y \in A} d(x, y)$.

Definition 1.1. *Let* (X,d) *be a metric space. The application* $h : \mathcal{K}(X) \times \mathcal{K}(X) \to [0,+\infty)$ *defined by* $h(A,B) = \max(d(A,B), d(B,A))$, *where* $d(A,B) = \sup_{x\in A} d(x,B) = \sup_{x\in A}(\inf_{y\in B} d(x,y))$ *is called the Hausdorff-Pompeiu metric.*

Remark 1.1. [1] $(K(X), h)$ *is a complete metric space if* (X,d) *is a complete metric space, compact if* (X,d) *is compact and separable if* (X,d) *is separable.*

Proposition 1.1. [1] *Let* (X, d_X) *and* (Y, d_Y) *two metric spaces. Then:*

1) If H *and* K *are two nonempty subsets of* X *then* $h_X(H,K) = h_X(\overline{H}, \overline{K})$, *where* h_X *is the the Hausdorff-Pompeiu semidistance associated to distance* d_X.

2) If $(H_i)_{i\in I}$ *and* $(K_i)_{i\in I}$ *are two families of nonempty subsets of* X *then:*

$$h_X(\underset{i\in I}{\cup} H_i, \underset{i\in I}{\cup} K_i) = h_X(\overline{\underset{i\in I}{\cup} H_i}, \overline{\underset{i\in I}{\cup} K_i}) \le \sup_{i\in I} h_X(H_i, K_i).$$

3) If H *and* K *are two nonempty subsets of* X *and* $f : X \to Y$ *a function then:*

$$h_Y(f(K), f(H)) \le Lip(f) \cdot h_X(K,H).$$

4) If $(H_n)_{n\geq 1} \subset P(X)$ *is a sequence of sets of* X *and* $H \in P(X)$ *is a set such that* $h_X(H, H_n) \to 0$, *then a element* $x \in X$ *belongs to* H *if and only if there exists* $x_n \in H_n$, $\forall\, n \geq 1$ *such that* $x_n \to x$.

5) If $(H_n)_{n\geq 1} \subset P(X)$ *is a sequence of relatively compact sets and* $H \in P(X)$ *is a set such that* $h_X(H, H_n) \to 0$, *then* H *is a relatively compact set.*

6) If $(H_n)_{n\geq 1} \subset P(X)$ *is a sequence of compact connected sets and* $H \in P(X)$ *is a closed set such that* $h_X(H, H_n) \to 0$, *then* H *is a compact connected set.*

Definition 1.2. *Let* (X,d) *be a metric space. For a function* $f : X \to X$ *let us denote by* $Lip(f) \in [0,+\infty]$ *the Lipschitz constant associated to f which is* $Lip(f) = \sup_{x,y\in X\ ;\ x\neq y} \dfrac{d(f(x), f(y))}{d(x,y)}$.

We say that f *is a Lipschitz function if* $Lip(f) < +\infty$ *and a contraction if* $Lip(f) < 1$.

Definition 1.3. *An iterated function system (IFS) on a metric space* (X,d) *consists in a finite family of contractions* $(f_k)_{k=\overline{1,n}}$ *on* X *and it is denoted by* $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$.

Definition 1.4. *For an IFS,* $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$, *the function* $F_{\mathcal{S}} : \mathcal{K}(X) \to \mathcal{K}(X)$ *defined by* $F_{\mathcal{S}}(B) = \overset{n}{\underset{k=1}{\cup}} f_k(B)$ *is called the fractal operator associated with the IFS* $\mathcal{S}$.

Proposition 1.2. *The function* $F_{\mathcal{S}}$ *is a contraction satisfying* $Lip(F_{\mathcal{S}}) \le \max_{k=\overline{1,n}} Lip(f_k)$.

Proof: Using proposition 1.1. we have for K, $K_1 \in \mathcal{P}(X)$ that:

$$h(F_{\mathcal{S}}(K), F_{\mathcal{S}}(K_1)) = h(\bigcup_{k=1}^{n} f_k(K), \bigcup_{k=1}^{n} f_k(K_1)) \le \max_{k=\overline{1,n}} h(f_k(K), f_k(K_1)) \le$$
$$\max_{k=\overline{1,n}} (Lip(f_k) \cdot h(K, K_1)) = h(K, K_1) \cdot \max_{k=\overline{1,n}} Lip(f_k). \qquad (1.1.)$$

For an IFS $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$, by Banach's contraction theorem, there exists an unique set $A(\mathcal{S})$ such that $F_{\mathcal{S}}(A(\mathcal{S})) = A(\mathcal{S})$, which is called the *attractor* of the IFS $\mathcal{S}$. More precisely we have the following well-known result.

Theorem 1.1. [4] *Let (X, d) be a complete metric space and $S = (X, (f_k)_{k=\overline{1,n}})$ an IFS with $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$. Then there exists an unique set $A(S) \in K(X)$ such that $F_{\mathcal{S}}(A(S)) = A(S)$. Moreover, for any $H_0 \in K(X)$ the sequence $(H_n)_{n\ge1}$ defined by $H_{n+1} = F_{\mathcal{S}}(H_n)$ is convergent to $A(S)$. For the speed of the convergence we have the following estimation $h(H_n, A(S)) \le \frac{c^n}{1-c} h(H_0, H_1)$.*

In the followings we briefly present the shift space of an IFS. For more details one can see [2]. We start with some set notations: $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}^* = \mathbb{N} - \{0\}$, $\mathbb{N}_n^* = \{1, 2, ..., n\}$. For two nonempty sets A and B, B^A denotes the set of functions from A to B. By $\Lambda = \Lambda(B)$ we will understand the set $B^{\mathbb{N}^*}$ and by $\Lambda_n = \Lambda_n(B)$ we will understand the set $B^{\mathbb{N}_n^*}$. The elements of $\Lambda = \Lambda(B) = B^{\mathbb{N}^*}$ will be written as infinite words $\omega = \omega_1\omega_2...\omega_m\omega_{m+1}...$, where $\omega_m \in B$ and the elements of $\Lambda_n = \Lambda_n(B) = B^{\mathbb{N}_n^*}$ will be written as finite words $\omega = \omega_1\omega_2...\omega_n$. By λ we will understand the empty word. Let us remark that $\Lambda_0(B) = \{\lambda\}$. By $\Lambda^* = \Lambda^*(B)$ we will understand the set of all finite words $\Lambda^* = \Lambda^*(B) = \bigcup_{n\ge0} \Lambda_n(B)$. We denote by $|\omega|$ the length of the word ω. An element of $\Lambda = \Lambda(B)$ is said to have length $+\infty$. If $\omega = \omega_1\omega_2...\omega_m\omega_{m+1}...$ or if $\omega = \omega_1\omega_2...\omega_n$ and $n \ge m$ then $[\omega]_m := \omega_1\omega_2...\omega_m$. More generally if $l < m$, $[\omega]_m^l = \omega_{l+1}\omega_{l+2}...\omega_m$ and we have $[\omega]_m = [\omega]_l[\omega]_m^l$ for $\omega \in \Lambda_n(B)$ if $n \ge m > l \ge 1$ and for $\omega \in \Lambda(B)$ if $m > l \ge 1$. For two words $\alpha, \beta \in \Lambda^*(B) \cup \Lambda(B)$, $\alpha \prec \beta$ means $|\alpha| \le |\beta|$ and $[\beta]_{|\alpha|} = \alpha$. For $\alpha \in \Lambda_n(B)$ and $\beta \in \Lambda_m(B)$ or $\beta \in \Lambda(B)$, by $\alpha\beta$ we will understand the joining of the words α and β namely $\alpha\beta = \alpha_1\alpha_2...\alpha_n\beta_1\beta_2...\beta_m$ and respectively $\alpha\beta = \alpha_1\alpha_2...\alpha_n\beta_1\beta_2...\beta_m\beta_{m+1}...$. On $\Lambda = \Lambda(\mathbb{N}_n^*) = (\mathbb{N}_n^*)^{\mathbb{N}^*}$ we can consider the metric $d_s(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{1-\delta_{\alpha_k}^{\beta_k}}{3^k}$ where $\delta_x^y = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \ne y \end{cases}$, $\alpha = \alpha_1\alpha_2...$ and $\beta = \beta_1\beta_2\ldots$

Let (X, d) be a complete metric space, $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ an IFS on X and $A = A(\mathcal{S})$ the attractor of the IFS $\mathcal{S}$. For $\omega = \omega_1\omega_2...\omega_m \in \Lambda_m(\mathbb{N}_n^*)$, f_ω denotes $f_{\omega_1} \circ f_{\omega_2} \circ ... \circ f_{\omega_m}$ and H_ω denotes $f_\omega(H)$ for a set $H \subset X$. By H_λ we will understand the set H. In particular $A_\omega = f_\omega(A)$.

The main results concerning the relation between the attractor of an IFS and the shift space is contained in the following theorem.

Theorem 1.2. [4] *Let (X, d) be a complete metric space. If $A = A(S)$ is the attractor of the IFS $S = (X, (f_k)_{k=\overline{1,n}})$ then:*

1) For $\omega \in \Lambda = \Lambda(N_n^*)$, $A_{[\omega]_{m+1}} \subset A_{[\omega]_m}$ and $d(A_{[\omega]_m}) \to 0$ when $m \to \infty$; more precisely

$$d(A_{[\omega]_m}) \leq c^m d(A).$$

2) If a_ω is defined by $\{a_\omega\} = \bigcap_{m\geq 1} A_{[\omega]_m}$ then $d(e_{[\omega]_m}, a_\omega) \to 0$ when $m \to \infty$, where $e_{[\omega]_m}$ is the unique fixed point of $f_{[\omega]_m}$.

3) $A = A(S) = \bigcup_{\omega\in\Lambda} \{a_\omega\}$, $A_\alpha = \bigcup_{\omega\in\Lambda} \{a_{\alpha\omega}\}$ for every $\alpha \in \Lambda^*$, $A = \bigcup_{\omega\in\Lambda_m} A_\omega$ for every $m \in N^*$ and more general $A_\alpha = \bigcup_{\omega\in\Lambda_m} A_{\alpha\omega}$ for every $\alpha \in \Lambda^*$ and every $m \in N^*$.

4) The set $\{e_{[\omega]_m} \mid \omega \in \Lambda$ and $m \in N^*\}$ is dense in A.

5) The function $\pi : \Lambda \to A$ defined by $\pi(\omega) = a_\omega$ is continuous and surjective.

Definition 1.5. *The function $\pi : \Lambda \to A = A(\mathcal{S})$ from the theorem 2.1. is called the canonical projection from the shift space on the attractor of the IFS $\mathcal{S}$.*

2. **Connectedness of the attractors of IFS**

Theorem 2.1. [5] *Let (X, d) be a complete metric space and $S = (X, (f_k)_{k=\overline{1,n}})$ an IFS with $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$. Let $A_0 \in K(X)$, $A_m = F_{\mathcal{S}}^{[m]}(A_0)$ such that $A_0 \subset F_{\mathcal{S}}(A_0)$. Then $A_m \subset A_{m+1}$ and $A(S) = \overline{\bigcup_{m\geq 1} A_m}$.*

Example 2.1. We consider the set $\mathbb{R}$ endowed with the distance given by the absolute value and the functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = \frac{x}{3}$ and $f_2(x) = \frac{x}{3} + \frac{2}{3}$ and the IFS $S = (\mathbb{R}, \{f_1, f_2\})$. Then $A(S) = C =$ the Cantor set.

Indeed, let $A_0 = E_0 = [0, 1]$ and $E_n = F_{\mathcal{S}}^{[n]}(E_0)$. We have $E_1 = F_{\mathcal{S}}(E_0) = f_1(E_0) \cup f_2(E_0) = [0, \dfrac{1}{3}] \cup [\dfrac{2}{3}, 1] = A_1$.

We prove by mathematical induction that $E_n = A_n$, for every $n \in \mathbb{N}$.

For $n = 0$ we have $E_0 = A_0$. We suppose now that $E_n = A_n$. Then:

$$E_{n+1} = F_{\mathcal{S}}(E_n) = F_{\mathcal{S}}(A_n) = F_{\mathcal{S}}(\bigcup_{i=0}^{2^n-1} [a_i^n, a_i^n + \frac{1}{3^n}]) =$$
$$\bigcup_{i=0}^{2^n-1} (f_1([a_i^n, a_i^n + \frac{1}{3^n}]) \cup f_2([a_i^n, a_i^n + \frac{1}{3^n}])) =$$
$$\bigcup_{i=0}^{2^n-1} ([a_i^{n+1}, a_i^{n+1} + \frac{1}{3^{n+1}}] \cup [a_{2^n+i}^{n+1}, a_{2^n+i}^{n+1} + \frac{1}{3^{n+1}}]) =$$
$$\bigcup_{i=0}^{2^{n+1}-1} [a_i^{n+1}, a_i^{n+1} + \frac{1}{3^{n+1}}] = A_{n+1}. \qquad (2.1.)$$

Thus, we obtained $E_n = A_n \to A(S)$. But $\bigcap_{n\geq 1} A_n = C$, so $A(S) = C$.

Definition 2.1. *Let X be a nonempty set and $(A_i)_{i\in I}$ a family of nonempty subsets of X. Then the family $(A_i)_{i\in I}$ is called connected if for* *every* $i, j \in I$ *there exists* $(i_k)_{k=\overline{1,n}} \subset I$ *such that* $i_1 = i$, $i_n = j$ *and* $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$, *for every* $k \in \{1, ..., n-1\}$.

Definition 2.2. *A metric space* (X, d) *is called arcwise connected if for every* $x, y \in X$ *there exists a continuous function* $\varphi : [0,1] \to X$ *such that* $\varphi(0) = x$ *and* $\varphi(1) = y$.

Definition 2.3. *A metric space* (X, d), *is called locally arcwise connected if every point has a neighbourhood that is arcwise connected.*

Lemma 2.1. [3] *Let* (X, d) *be a complete metric space and* $S = (X, (f_i)_{i=\overline{1,n}})$ *an IFS with* $c = \max_{i=\overline{1,n}} Lip(f_i) < 1$. *We also consider* $A(S)$ the attractor of S. *Let* $u : [0,1] \to A(S)$ *and for* $t \in [0,1]$ *we define:*

$$D(u, t) = \sup\{\limsup_{n\to\infty} d(u(t_n), u(s_n)) \mid \lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = t\}.$$

If $f_n : [0,1] \to A(S)$ *converges uniformly to* $f : [0,1] \to A(S)$ *when* $n \to \infty$ *and* $D(f_n, s) = 0$, *then* f *is continuous in* s.

Proof: Let d be a metric on $A(S)$ compatible with the topology of $A(S)$. If $t_n \overset{n\to\infty}{\to} s$ and $s_n \overset{n\to\infty}{\to} s$ then:

$$d(f(t_n), f(s_n)) \leq \\ d(f(t_n), f_m(t_n)) + d(f_m(t_n), f_m(s_m)) + d(f_m(s_m), f(s_m)). \qquad (2.2.)$$

Let $r_m = \sup\{d(f_m(t), f(t)) \mid t \in [0,1]\}$. The above inequality becomes $D(f, s) \leq 2r_m + D(f_m, s)$.

We let $m \to \infty$ and we obtain $D(f, s) = 0$. Thus f is continuous in s.

In what concerns the connectedness of the attractor of an IFS with have the following theorem:

Theorem 2.2. [3] *Let* (X, d) *be a complete metric space and* $S = (X, (f_k)_{k=\overline{1,n}})$ *an IFS with* $c = \max_{i=\overline{1,n}} Lip(f_i) < 1$. *We also consider* $A(S)$ *the attractor of* S *and we denote by* $A_i = f_i(A(S))$, *for every* $i \in \{1, \ldots, n\}$. *Then we have the equivalence:*

1) The falimy $(A_i)_{i=\overline{1,n}}$ *is connected.*

2) $A(S)$ *is arcwise connected.*

3) $A(S)$ *is connected.*

We will apply the theorem 2.2. to some examples to find out whether or not the attractor is connected.

Example 2.2. We consider the set $\mathbb{R}$ endowed with the distance given by the absolute value and the functions f_1,$f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = \frac{x}{2}$ and $f_2(x) = \frac{x}{2} + \frac{1}{2}$ and the IFS $S = (\mathbb{R}, \{f_1, f_2\})$. Then $A(S) = [0,1]$.

Indeed, $F_{\mathcal{S}}([0,1]) = f_1([0,1]) \cup f_2([0,1]) = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1] = [0,1]$.

We have $A_1 = f_1([0,1]) = [0, \frac{1}{2}]$, $A_2 = [\frac{1}{2}, 1]$ şi $A_1 \cap A_2 = \{\frac{1}{2}\}$.

We remark that $A(S)$ is connected and the family (A_1, A_2) is also connected.

Example 2.3. We consider the set $\mathbb{R}$ endowed with the distance given by the absolute value and the functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = \frac{x}{3}$

and $f_2(x) = \frac{x}{3} + \frac{2}{3}$ and the IFS $S = (\mathbb{R}, \{f_1, f_2\})$. Then $A(S) = C \subset [0,1]$ is called the Cantor set.

We have that $A_1 \subset [0, \frac{1}{3}]$, $A_2 \subset [\frac{2}{3}, 1]$ and $A_1 \cap A_2 = \emptyset$. Thus the family (A_1, A_2) is not connected and so $A(S)$ is not a connected set.

Example 2.4. We consider the set $\mathbb{R}^2$ endowed with the euclidean distance and the functions $f_1, f_2, f_3 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f_1(x,y) = (\frac{x}{2}, \frac{y}{2})$, $f_2(x,y) = (\frac{x}{2}+\frac{1}{2}, \frac{y}{2})$ and $f_3(x,y) = (\frac{x}{2}+\frac{1}{2}, \frac{y}{2}+\frac{1}{2})$ and the IFS $S = (\mathbb{R}, (f_1, f_2, f_3))$. The functions f_1, f_2, f_3 are similitudes of the plane with the coefficient $\frac{1}{2}$. The attractor $A(S)$ is called the Sierpinski triangle.

We have $f_1(0,0) = (0,0)$, $f_2(1,0) = (1,0)$ and $f_3(0,1) = (0,1)$. So $(0,0), (1,0), (0,1) \in A(S)$. Also:

$$\begin{aligned} f_1(1,0) = f_2(0,0) = (\tfrac{1}{2}, 0) \in A_1 \cap A_2, \\ f_1(0,1) = f_3(0,0) = (0, \tfrac{1}{2}) \in A_1 \cap A_3, \\ f_2(0,1) = f_3(1,0) = (\tfrac{1}{2}, \tfrac{1}{2}) \in A_2 \cap A_3. \end{aligned} \qquad (2.3.)$$

It results that the family of sets (A_1, A_2, A_3) is connected and thus $A(S)$ is connected and arcwise connected.

Next we wil study the case of the attrators with many connected components. We have a result similar to theorem 2.2.

Theorem 2.3. [7] *Let (X,d) be a complete metric space, $p \in N^*$, $S = (X, (f_k)_{k=\overline{1,n}})$ an IFS with $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$ and $A(S)$ the attractor of S. Then we have the equivalence:*

1) For every $\omega \in \Lambda_p = \Lambda_p(N_n^*)$ the family $(A_{\omega i})_{i=\overline{1,n}}$ is connected.

2) The set A_ω is arcwise connected for every $\omega \in \Lambda_p$.

3) The set A_ω is connected for every $\omega \in \Lambda_p$.

4) The set A_ω is arcwise connected for every $\omega \in \Lambda_m$ and $m \geq p$.

5) The set A_ω is connected for every $\omega \in \Lambda_m$ and $m \geq p$.

Moreover, if one of the conditions from 1)-5) is fulfilled then we have the following:

6) $A(S)$ has at most n^p connected components, moreover, $A(S)$ has the same number of connected components as the family $(A_\omega)_{\omega \in \Lambda_p}$.

7) Each connected component of $A(S)$ is arcwise connected.

8) $A(S)$ is locally arcwise connected.

Example 2.5. We consider the function $\phi : [0,1] \to [0,1]$ defined by:

$$\phi(x) = \begin{cases} \dfrac{x}{2}, & \text{if } x \in [0, \dfrac{1}{4}] \\ \dfrac{1}{8}, & \text{if } x \in [\dfrac{1}{4}, \dfrac{3}{4}] \\ \dfrac{x}{2} - \dfrac{1}{4}, & \text{if } x \in [\dfrac{3}{4}, 1] \end{cases} .$$

Then $Lip(\phi) = \dfrac{1}{2}$.

Let $\psi : [0,1] \to [0,1]$ a function defined by $\psi(x) = 1 - \phi(1-x)$. Then $Lip(\psi) = \dfrac{1}{2}$.

We consider the iterated function system $S = ([0,1], \{\phi, \psi\})$. Then $A(S) = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$.

Indeed,

$$\phi([0,\frac{1}{4}] \cup [\frac{3}{4},1]) = \phi([0,\frac{1}{4}]) \cup \phi([\frac{3}{4},1]) = [0,\frac{1}{8}] \cup [\frac{1}{8},\frac{1}{4}] = [0,\frac{1}{4}] \text{ and}$$
$$\psi([0,\frac{1}{4}] \cup [\frac{3}{4},1]) = \psi([0,\frac{1}{4}]) \cup \psi([\frac{3}{4},1]) =$$
$$(1-\phi([\frac{3}{4},1])) \cup (1-\phi([0,\frac{1}{4}])) = [\frac{3}{4},\frac{7}{8}] \cup [\frac{7}{8},1] = [\frac{3}{4},1]. \quad (2.4.)$$

Thus $F_{\mathcal{S}}([0,\frac{1}{4}]\cup[\frac{3}{4},1]) = \phi([0,\frac{1}{4}]\cup[\frac{3}{4},1])\cup\psi([0,\frac{1}{4}]\cup[\frac{3}{4},1]) = [0,\frac{1}{4}]\cup[\frac{3}{4},1]$.

It results that $A(S) = [0,\frac{1}{4}] \cup [\frac{3}{4},1]$.

We have $A_1 = \phi(A) = [0,\frac{1}{4}]$, $A_2 = \psi(A) = [\frac{3}{4},1]$, $A_{11} = [0,\frac{1}{8}]$, $A_{12} = [\frac{1}{8},\frac{1}{4}]$, $A_{21} = [\frac{3}{4},\frac{7}{8}]$ and $A_{22} = [\frac{7}{8},1]$. Then $A_1 \cap A_2 = \emptyset$, $A_{11} \cap A_{12} = \{\frac{1}{8}\}$ and $A_{21} \cap A_{22} = \{\frac{7}{8}\}$. That means the families of set (A_{11}, A_{12}) and (A_{21}, A_{22}) are connected. We remark that A_1 and A_2 are connected sets, but $A(S)$ is not connected.

We will study the case infinite iterated function systems (IIFS). The theorem 2.2 is not true in that case as one can see from the above example.

Example 2.6. Let (X,d) be a complete metric space and $A \in \mathcal{B}(X)$ a bounded set. For an element $a \in X$, f_a will mean the constant function a, i.e. $f_a : X \to X$ and $f_a(x) = a$ for every $x \in X$. Then $A(S)$ is the attactor of IIFS $S = (X, (f_a)_{a\in A})$, if A is infinite, or IFS $S = (X, (f_a)_{a\in A})$, if A is finite. Also, A is the attractor of IIFS $S_B = (X, (f_a)_{a\in B})$ for every dense set B in A. In particular, that happens for every compact set A. Because a compact set can be connected, but not arcwise connected, it follows that 2) and 3) from the theorem 2.1 are not equivalent in general for an IIFS.

On the other hand, because the family $(A_a = f_a(A) = \{a\})_{a\in A}$ is not connected for every set A, in fact $A_a \cap A_b = \emptyset$, for $a \neq b$, we also obtain that 1) and 2) from theorem 4.1. are not equivalent in general for an IIFS.

We give some sufficient conditions for an attractor of IIFS to be connected.

Theorem 2.4. [6] *Let (X,d) be a complete metric space, $S = (X, (f_i)_{i\in I})$ an IIFS and $A(S)$ the attractor of S. Let $I_j \subset I$, for every $j \in J$ such that:*

1) $I = \bigcup_{j\in J} I_j$,

2) $\bigcup_{j\in J} B_j$ is a connected set, where $B_j := A(S_j)$ is the attractor of $S_j = (X, (f_i)_{i\in I_j})$, for every $j \in J$.

Then $A(S)$ is a connected set.

Corolarry 2.1. *Let (X,d) be a complete metric space and $S = (X, (f_i)_{i\in I})$ an IIFS. Let $I_j \subset I$, for every $j \in J$ such that:*

1) $I = \bigcup_{j \in J} I_j$.

2) B_j *is connected, where* $B_j := A(S_j)$ *is the attractor of* $S_j = (X, (f_i)_{i \in I_j})$ *for every* $j \in J$.

3) The family $(B_j)_{j \in J}$ *is connected.*

Then $A(S)$ *is connected.*

Proof: Because $(B_j)_{j \in J}$ is a connected family of connected sets it results that $\bigcup_{j \in J} B_j$ is connected. Hence, from theorem 2.4 $A(S)$ is connected.

Corolarry 2.2. *Let* (X, d) *be a complete metric space and* $S = (X, (f_i)_{i \in I})$ *an IIFS. Let* $I_j \subset I$, *for every* $j \in J$ *such that:*

1) $I = \bigcup_{j \in J} I_j$.

2) I_j *is finite for every* $j \in J$.

3) The families of sets $(f_i(B_j))_{i \in I_j}$ *are connected, where* $B_j := A(S_j)$ *is the attractor of* $S_j = (X, (f_i)_{i \in I_j})$ *for every* $j \in J$.

4) The family $(B_j)_{j \in J}$ *is connected.*

Then $A(S)$ *is connected.*

Proof: Let $j \in J$. Because I_j is finite then $S_j = (X, (f_i)_{i \in I_j})$ is an IFS. Because the family $(f_i(B_j))_{i \in I_j}$ is connected, where $B_j = A(S_j)$ is the attractor of S_j, then from theorem 2.2 B_j is connected, for every $j \in J$, because j was chosen arbitrarly. Thus, from corolarry 2.1, $A(S)$ is connected.

References

1. Barnsley, M.F., *Fractals Everywhere*, "Academic Press", New York, 1998.
2. Falconer, K.J., *Fractal Geometry-Foundations and Applications*, "John Wiley", 1990.
3. Kigami, J., *Analysis on Fractals*, "Cambridge Univ. Press", 2001.
4. Secelean, N.A., *Măsură şi Fractali*, "Editura Univ. Lucian Blaga", Sibiu, 2002.
5. Edgar, G. *Measure, Topology and Fractal Geometry*, "Springer", 2008.
6. Dumitru, D., and Mihail., A., *A Sufficient Condition for the Connectedness of the Attractors of Infinite Iterated Function Systems,* "An. Stiint. Univ. Al. I. Cuza Iasi Mat.", 55, suppl. 1, 87-94, 2009.
7. Mihail, A., *On the Connectivity of Attractors of Iterated Multifunction Systems,* "Real Analysis Exchange", 34, 195-206, 2008-2009.
8. Hutchinson, J., *Fractals and Self-similarity*, "Indiana Univ. J. Math." 30, 1981.