A PERRON-FROBENIUS THEOREM FOR NONLINEAR OPERATORS

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Abstract
We extend a result of T. Fujimoto and U. Krause to infinite dimensional spaces, using a notion of strongly increasing operators which works also in spaces having the interior of the positive cone empty.

Key-words: *Perron-Frobenius theorem, nonlinear operators, Banach lattices*

AMS classification: 05C38, 15A15, 05A15, 15A18

1. Introduction

We will use some notions and results from the theory of Banach lattices. In this paper, E denotes a real vector lattice. We denote by $x \vee y$ the supremum of the set $\{x, y\}$, and by $x \wedge y$ the infimum of the same set. The modulus of an element x is defined by $|x| = x \vee (-x)$. Also, we denote by E_+ the set of positive elements of E.

For the order relation between vectors, we use the notation "$\leqq$", while for real numbers we use the notation "$\leq$". For the strict relations, we use, respectively, "$\lneqq$" and "$<$".

An ideal of E is a subspace of E, denoted I, having the property that from $x \in I$, and $|y| \leqq |x|$, it follows $y \in I$. The ideal generated by an element $a \in E_+$ is the set:

$$I_a = \{x \in E : \exists r \in \mathbb{R} \text{ such that} |x| \leqq r.|a|\}$$

If the vector lattice E is in the same time a Banach space, and if from $|x| \leqq |y|$ it follows $\|x\| \leq \|y\|$, then E is called a Banach lattice.

Let $v \in E$ be any vector. A sequence $(x_n)_{n\in\mathbb{N}}$ is called v-*convergent* to $x \in E$ if for every $\varepsilon \in \mathbb{R},\ \varepsilon>0$, there is $n_\varepsilon \in \mathbb{N}$ such that $n \geq n_\varepsilon$ implies $|x_n - x| \leqq \varepsilon.v.$ The sequence $(x_n)_{n\in\mathbb{N}}$ is called (*)-*convergent* to x if every subsequence of it has a subsequence which is v-*convergent* to x, for some $v \in E$, depending on this subsequence. The following result is known ([1]). We give also the proof, because we use it later.

Lemma 1. *Let E be a Banach lattice. The following conditions are equivalent:*
i) The sequence $(x_n)_{n\in\mathbb{N}}$ is norm convergent to x.
ii) The sequence $(x_n)_{n\in\mathbb{N}}$ is (*)-*convergent to x.*

Proof. If $\lim\limits_{n\to\infty}\|x_n\| = 0$, then there is a strictly-increasing sequence $(j_n)_{n\in\mathbb{N}}$ of natural numbers, such that $\lim\limits_{n\to\infty} n^3 \left\|x_{j_n}\right\| = 0$. If we put $v = \sum\limits_{n\geq 1} n\left|x_{j_n}\right|$, then we have $\left|x_{j_n}\right| \leqq \frac{1}{n} v$, for every $n \in \mathbb{N}$, and so $\left(x_{j_n}\right)_{n\in\mathbb{N}}$ is v-*convergent* to x. Because in the place of the sequence $(x_n)_{n\in\mathbb{N}}$ we can take any subsequence, it follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is (*)-*convergent* to x.

Conversely, if the sequence $(x_n)_{n\in\mathbb{N}}$ is (*)-*convergent* to 0, then for every subsequence of the sequence $(x_n)_{n\in\mathbb{N}}$, there is a subsequence which is v-*convergent* to 0. Because the norm is monotone, this subsequence is norm convergent to x.

Remark 1. *In a Banach lattice the positive cone is closed, and this implies (see [1]):*
i) $x_n \leqq y_n \Rightarrow \lim\limits_{n\to\infty} x_n \leqq \lim\limits_{n\to\infty} x_n$;
ii) $x \in E_+ \Rightarrow \inf\left\{\frac{1}{n}x \text{ :n} \in \mathbb{N}\right\} = 0$.

Definition 1 *(See [3]). Let E, F be real vector spaces, and let $C \subseteq E$ be a cone, not necessarily convex.*
An operator $A : C \to F$ is called *weak-homogeneous* if there is a function $k : \mathbb{R}_+ \to \mathbb{R}_+$ such that:
i) $k(0) = 0$;

ii) $\forall \lambda_1, \lambda_2 \in \mathbb{R}_+$, $\lambda_1 \leq \lambda_2 \Rightarrow \dfrac{k(\lambda_1)}{\lambda_1} \geqq \dfrac{k(\lambda_2)}{\lambda_2}$;

iii) $A(\lambda x) = k(\lambda) A(x)$, $\forall \lambda \in \mathbb{R}_+$, $\forall x \in C$.

The operator A is called *ray-preserving* if:

$$\forall x \in C,\ \forall \lambda \in \mathbb{R}_+, \exists \lambda' \in \mathbb{R}_+ \text{ such that } A(\lambda . x) = \lambda' A(x).$$

Remark 2. *If A is not identically* 0*, it follows also:*

1) $k\left(\dfrac{1}{\lambda}\right) = \dfrac{1}{k(\lambda)}$, $\forall \lambda \in \mathbb{R}_+$, $\lambda \neq 0$;

2) $k(1) = 1$.

In the economic theory, we deal with nonlinear operators of the following type. Let $A_i : C \to F$, $i=\overline{1,...,n}$, be linear operators taking values in the vector lattice F. We define $A : C \to F$ by $A(x) = \inf_i A_i(x)$.

Then A is a homogeneous operator, which is not linear. From the point of view of economic equilibrium theory, we are interested in the ergodic behaviour of such nonlinear operators.

2. **The main result**

One of the difficulties encountered in working in infinite dimensional ordered spaces is the fact that for the majority of the usual spaces, the topological interior of the positive cone is empty. Such kind of difficulties appears when we want, for example, to define the notion of strongly-increasing operator. Usually, this is done in the following way. An operator T is called *strongly-increasing* if:

$$\forall x, y \in E_+,\ 0 \leqq x \lneqq y \Rightarrow T(y) - T(x) \in \operatorname{int}(F_+).$$

If $\operatorname{int}(F_+) = \varnothing$, we can use the following definition.

Definition 2. *Let E, F be ordered vector spaces, $T : E \to F$ an operator. We say that T is strongly-increasing if:*

$$\forall x, y \in E_+,\ 0 \leqq x \lneqq y \Rightarrow \exists \varepsilon \in (0,1) \text{ a.i. } T(x) \lneqq \varepsilon T(y).$$

The following theorem is the fundamental result of this paper. It was proved for the finite dimensional case in [3].

Theorem 1. *Let E be a real Banach lattice, E_+ the positive cone, and $T: E_+ \to E_+$ an weak-homogeneous and complete-continuous operator (this means it is continuous, and transform bounded sets in relatively-compact ones). Let $x^0 \in E_+$ be such that I_{x^0} (the ideal generated by x^0) is closed, and that:*

$$I_{x^o} = I_{T(x)}, \ \forall \mathrm{x} \in I_{x^o}, \ \mathrm{x} \gneqq 0.$$

We suppose that T is strongly-increasing. Then there is an unique $x^0 \in I_{x^0}$ with $\left\|x^0\right\| = 1$ and an unique $\lambda^* \in \mathbb{R}_+, \ \lambda^* > 0$, such that the sequence:

$$x^n = \frac{T^n\left(x^0\right)}{\left\|T^n\left(x^0\right)\right\|}$$

is convergent to x^*, and:

$$T\left(x^*\right) = \lambda^* x^*.$$

Proof. If $x \neq 0, \ \mathrm{x} \in \mathrm{E}_+$, the fact that T is strongly-increasing implies $T\left(x\right) \neq 0$, hence we can define the sequence:

$$x^n = \frac{T^n\left(x^0\right)}{\left\|T^n\left(x^0\right)\right\|}, \ \mathrm{n} \in \mathbb{N}, \mathrm{n} \neq 0.$$

Because T is weak-homogeneous, we have:

$$x^{n+1} = \frac{T^{n+1}\left(x^0\right)}{\left\|T^{n+1}\left(x^0\right)\right\|} = \frac{T\left(\dfrac{T^n\left(x^0\right)}{\left\|T^n\left(x^0\right)\right\|}\left\|T^n\left(x^0\right)\right\|\right)}{\left\|T\left(\dfrac{T^n\left(x^0\right)}{\left\|T^n\left(x^0\right)\right\|}\left\|T^n\left(x^0\right)\right\|\right)\right\|} =$$

$$= \frac{k\left(\left\|T^n\left(x^0\right)\right\|\right)T\left(x^n\right)}{k\left(\left\|T^n\left(x^0\right)\right\|\right)\left\|T\left(x^n\right)\right\|} = \frac{T\left(x^n\right)}{\left\|T\left(x^n\right)\right\|}.$$

Thus, we have the following recurrence relation:

$$x^{n+1} = \frac{T\left(x^n\right)}{\left\|T\left(x^n\right)\right\|}.$$

Because $I_{T^n\left(x^0\right)} = I_{x^0}$ we can define:

$$\lambda_n = \min\left\{\kappa \in \mathbb{R}_+ :\ \mathrm{T}\left(\mathrm{x}^{\mathrm{n}}\right) \leqq \lambda x^n\right\},$$

for every $\mathrm{n} \in \mathbb{N}$. This sequence is nonincreasing. Indeed, if $\lambda \in \mathbb{R}_+$ is such that $\mathrm{T}\left(\mathrm{x}^{\mathrm{n}}\right) \leqq \lambda . x^n$, we have:

$$0 \lneqq \mathrm{T}\left(\mathrm{x}^{\mathrm{n}}\right) \Rightarrow \left\|\mathrm{T}\left(\mathrm{x}^{\mathrm{n}}\right)\right\| \leqq \lambda \left\|x^n\right\| = \lambda,$$

because the norm is monotone, and $\left\|x^n\right\| = 1$. Now because the function $\frac{k(\lambda)}{\lambda}$ is nonincreasing, we have:

$$T\left(x^{n+1}\right) = T\left(\frac{T\left(x^n\right)}{\left\|T\left(x^n\right)\right\|}\right) = \frac{1}{k\left\|T\left(x^n\right)\right\|} \cdot T\left(T\left(x^n\right)\right)$$

$$\leqq \frac{T\left(\lambda x^n\right)}{k\left(\left\|T\left(x^n\right)\right\|\right)} = \frac{k(\lambda)}{k\left(\left\|T\left(x^n\right)\right\|\right)} T\left(x^n\right)$$

$$= \frac{k(\lambda)}{\lambda} \cdot \frac{\lambda}{k\left(\left\|T\left(x^n\right)\right\|\right)} \cdot \frac{\left\|T\left(x^n\right)\right\|}{\left\|T\left(x^n\right)\right\|} T\left(x^n\right)$$

$$\leqq \frac{k\left(\left\|T\left(x^n\right)\right\|\right)}{\left\|T\left(x^n\right)\right\|} \cdot \frac{\lambda}{k\left(\left\|T\left(x^n\right)\right\|\right)} \cdot \frac{\left\|T\left(x^n\right)\right\|}{\left\|T\left(x^n\right)\right\|} T\left(x^n\right) = \lambda \frac{T\left(x^n\right)}{\left\|T\left(x^n\right)\right\|} = \lambda x^{n+1}.$$

It follows that $\lambda_{n+1} \leq \lambda$, hence $\lambda_{n+1} \leq \lambda_n$.

In the same way, putting:

$$\mu_n = \max\left\{\mu \in \mathbb{R}_+ :\ \mu \mathrm{x}_{\mathrm{n}} \leqq T\left(x_n\right)\right\},$$

we obtain a nonincreasing sequence of real numbers, such that

$$\mu_n \mathrm{x}^{\mathrm{n}} \leqq T\left(x_n\right) \leqq \lambda_n \mathrm{x}^{\mathrm{n}},\ \forall \mathrm{n} \in \mathbb{N}.$$

Because the norm is monotone, we have $\mu_n \leq \lambda_n$, for every $\mathrm{n} \in \mathbb{N}$.

Let $\mu^* = \lim\limits_n \mu_n$, and $\lambda^* = \lim\limits_n \lambda_n$. Then $\mu_n > 0$, hence $\mu > 0$. Because we have $\left\|x^n\right\| = 1$ and T is complete-continuous, there is a subsequence $\left(x^{n_j}\right)_{j \in \mathbb{N}}$ of the sequence $\left(x^n\right)_n$, with the property that the sequence $\left(T\left(x^{n_j}\right)\right)_j$ is convergent. We have $\mu_{n_j} x^{n_j} \leqq T\left(x_{n_j}\right) \leqq \lambda_{n_j} x^{n_j}$, hence from $\|x\|=1$ and because the norm is monotone, we obtain:

$$\mu_{n_j} \leqq \left\|T\left(x_{n_j}\right)\right\| \leqq \lambda_{n_j},$$

hence:

$$\mu_{n_j} x^{n_j} \leqq \left\|T\left(x_{n_j}\right)\right\|, \forall j \in \mathbb{N}.$$

This means that 0 is not a limit point of the sequence $\left(\left\|T\left(x^{n_j}\right)\right\|\right)_j$, and so the sequence:

$$x^{n_j+1} = \frac{T\left(x^{n_j}\right)}{\left\|T\left(x^{n_j}\right)\right\|}$$

is convergent to an element $x^* \in E_+$. Obviously, $\|x^*\|=1$ and $x^* \in I_0$, hence we have the inclusion $I_{x^*} \subseteq I_{x^0}$. Because $T\left(x^{n_j+1}\right) \leqq \lambda_{n_j+1} x^{n_j+1}$, and the positive cone is closed, it follows $T\left(x^*\right) \leqq \lambda^* x^*$. We will show that, in fact, we have equality.

First, we make a remark. If $I_a = I_b$, and $0 \leqq a \leqq s.b$, with $s \in (0,1)$ then:

$$b \geqq b-a = (1-s)b + sb - a \geqq (1-s)b,$$

hence $I_b = I_{b-a}$.

Let us suppose now that $T\left(x^*\right) \lneqq \lambda^* x^*$. Then, because T is strongly-increasing, it follows that there is $s \in (0,1)$ such that we have $T^2\left(x^*\right) \leqq sT\left(\lambda^* x^*\right)$. But we have also $0 \leqq \left\|T\left(x^*\right)\right\| \leqq \lambda^*$, hence we obtain the inequality:

$$\frac{k\left(\lambda^*\right)}{\lambda^*} \leq \frac{k\left(\left\|T\left(x^*\right)\right\|\right)}{\left\|T\left(x^*\right)\right\|}.$$

Thus,

$$T\left(\frac{T\left(x^*\right)}{\left\|T\left(x^*\right)\right\|}\right)=\frac{T^2\left(x^*\right)}{k\left\|T\left(x^*\right)\right\|}\leqq\frac{s.k\left(\lambda^*\right).T\left(x^*\right)}{k\left\|T\left(x^*\right)\right\|}$$

$$=\frac{s.k\left(\lambda^*\right).\lambda^*T\left(x^*\right)}{\lambda^*k\left(\left\|T\left(x^*\right)\right\|\right)}\leqq\frac{s.\lambda^*.k\left(\left\|T\left(x^*\right)\right\|\right)T\left(x^*\right)}{T\left(x^*\right)k\left(\left\|T\left(x^*\right)\right\|\right)}=\frac{s.\lambda^*T\left(x^*\right)}{\left\|T\left(x^*\right)\right\|}.$$

Let L be the limit of the sequence:

$$\frac{T\left(x^{n_j}\right)}{\left\|T\left(x^{n_j}\right)\right\|}\overset{not}{=}y_j.$$

Because T is continuous, we have:

$$L=\frac{T\left(x^*\right)}{\left\|T\left(x^*\right)\right\|}.$$

Also we have $x^*\in \mathrm{I}_{x^0}$, $x^*\gneqq 0$, hence by the hypothesis, $\mathrm{I}_{T\left(x^*\right)}=\mathrm{I}_{x^0}$. But $T\left(x^*\right)=L\left\|T\left(x^*\right)\right\|$, and so $\mathrm{I}_L=\mathrm{I}_{T\left(x^*\right)}=\mathrm{I}_{x^0}$.

Using the Lemma 1, let $\left(y_{j'}\right)$ be a subsequence of this sequence, which is v-*convergent* to L. Then $s.\lambda^*y_{j'}$ is v-*convergent to* $s.\lambda^*L$, and this means that for every $\varepsilon>0$ there is $j'_\varepsilon\in\mathbb{N}$, such that $\left|s.\lambda^*y_{j'}-s.\lambda^*.L\right|\leqq\varepsilon.v,\ \forall j'\geq j'_\varepsilon$ which implies:

$$s.\lambda^*.L-\varepsilon.v\leqq s.\lambda^*y_{j'}\leqq s.\lambda^*.L+\varepsilon.v,\ \forall j'\geq j'_\varepsilon. \tag{1}$$

Let now $\left(T\left(y_{j_i'}\right)\right)_i$ be a subsequence of the sequence $T\left(y_{j'}\right)$, which is w-*convergent* to $T\left(L\right)$. Then there is $i_\varepsilon\in\mathbb{N}$, such that for every $i\geq i_\varepsilon$, we have:

$$T\left(L\right)-\varepsilon.w\leqq T\left(y_{j'_i}\right)\leqq T\left(L\right)+\varepsilon.w. \tag{2}$$

Because from (1) we have $T\left(L\right)\leqq s.\lambda^*.L$, the preceding remark implies that $\mathrm{I}_{\lambda^*L-T(L)}=\mathrm{I}_{\lambda^*L}=\mathrm{I}_{x^0}$. The ideal I_{x^0} is closed, and the proof of Lemma 1 shows that we can take v and w from I_{x^0}, hence there is $\bar{\varepsilon}\in\mathbb{R},\ \bar{\varepsilon}>0$ such that:

$$\bar{\varepsilon}\left(v+w\right)\leqq s.\lambda^*L-T(L). \tag{3}$$

If we take $\bar{\varepsilon}=\varepsilon$ using (2), (3), then (1), we obtain:

$$T\left(y_{j'_i}\right) \leqq T(L) + \bar{\varepsilon}.w \leqq s.\lambda^* L - \bar{\varepsilon}.v \leqq \lambda^* y_{j'_i}$$
$$= s.\lambda^* \frac{T\left(x^{n_{j'_i}+1}\right)}{\left\| T\left(x^{n_{j'_i}+1}\right)\right\|} = s.\lambda^* x^{n_{j'_i}+2},$$

for i sufficiently large. From this, it follows $\lambda_{n_{j'_i}+2} \leqq s.\lambda^*$. But because $0 < s < 1$, this contradicts the fact that $\lambda^* \le \lambda_n,\ \forall n \in \mathbb{N}$. So, $T\left(x^*\right) = \lambda^* x^*$. Also, we obtain:

$$I_{x^*} = I_{T\left(x^*\right)} = I_{x^0}.$$

The uniqueness. Let $x \in E_+$ be such that $I_x = I_{x^0}$, and let $\lambda > 0$ be such that $T(x) = \lambda x$. We have:

$$T(x) = k\left(\|x\|\right) T\left(\frac{x}{\|x\|}\right),$$

so the condition $T(x) = \lambda.x$ is equivalent to:

$$T\left(\frac{x}{\|x\|}\right) = \frac{\lambda \|x\|}{k\left(\|x\|\right)} \frac{x}{\|x\|}.$$

This shows that, eventually taking $x' = \frac{x}{\|x\|}$ we can suppose $\|x\| = 1$. Also, due to the symmetry, we can take $\lambda \ge \lambda'$. Let $s = \max\left\{t \ge 0 :\ x^* - tx \geqq 0\right\}$, and let $z = x^* - sx$. Because $I_x = I_{x^*}$, there is $r > 0$ such that $rx^* \geqq x$, hence we have $s > \frac{1}{r} > 0$. This implies $x^* \geqq sx \gneqq 0$. The norm monotonicity gives $0 < s \le 1$, hence $\frac{k(s)}{s} \ge \frac{k(1)}{1} = 1$. From this, it follows:

$$T\left(x^*\right) = \lambda^* .x^* = \lambda^*\left(z + sx\right) \leqq \lambda^*\left(z + k(s).x\right)$$
$$= \lambda^*\left(.z + \frac{k(s)}{\lambda} T(x)\right) = \lambda^* z + \frac{\lambda^*}{\lambda} T(sx),$$

and so:

$$\lambda^* z \geqq T\left(x^*\right) - \frac{\lambda^*}{\lambda} T(sx).$$

If $z \ne 0$, then $x^* = z + sx \gneqq sx$, so there is $\mu_0 \in \mathbb{R},\ 0 < \mu_0 < 1$, such that

$$T\left(x^*\right) \gneqq \frac{1}{\mu} T\left(sx\right),\ \forall \mu \in \left[\mu_0, 1\right].$$

It follows:

$$\lambda^* z \geqq \left(\frac{1}{\mu} - \frac{\lambda^*}{\lambda}\right) T\left(sx\right),\ \forall \mu \in \left[\mu_0, 1\right].$$

Because $\lambda \geq \lambda^*$, and $\mu < 1$, it follows that $\left(\frac{1}{\mu} - \frac{\lambda^*}{\lambda}\right) > 0$, hence:

$$\mathrm{I}_{\mathrm{z}} \supseteq \mathrm{I}_{\mathrm{T(x)}}.$$

But $\mathrm{I}_{\mathrm{T(x)}} = \mathrm{I}_{\mathrm{x}^0} = \mathrm{I}_{\mathrm{x}}$, so there is $t \in \mathbb{R},\ t > 0$, such that $z - tx \geqq 0$. Hence:

$$0 \leqq z - tx = x^* - sx - tx = x^* - \left(s + t\right)x$$

which contradicts the choice of s. Hence $z = 0$, which means $x^* = sx$. Because $\left\|x^*\right\| = \left\|x\right\| = 1$, we obtain $s = 1,\ x^* = x$ and $\lambda^* = \lambda$.

We prove now the convergence of the sequence x^n. Let x^{n_j} be a subsequence of the sequence x^n. Because T is relatively-compact, the sequence $T\left(x^{n_j - 1}\right)$ has a convergent subsequence, say $T\left(x^{n_{j_k} - 1}\right)$. We have:

$$x^{n_{j_k}} = \frac{T\left(x^{n_{j_k} - 1}\right)}{\left\|T\left(x^{n_{j_k} - 1}\right)\right\|},$$

hence the subsequence $x^{n_{j_k}}$ is convergent to an element $x' \in I_{x^0}$ with $\left\|x'\right\| = 1$. The preceding reasoning applies again and we obtain the equality $T\left(x'\right) = \lambda' x'$, for some $\lambda' > 0$. The unicity implies $\lambda' = \lambda^*$, and $x' = x^*$. In conclusion, every subsequence of the sequence x^n has a subsequence which is convergent to x^*, and so the sequence x^n is convergent to x^*.

Example 1. *Let $K \subseteq \mathbb{R}^n$ be a compact set, and let $F : K \times K \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function. Let $X = C\left(K\right)$ the Banach lattice of continuous real functions defined on K, with the usual order. We define $T : X_+ \to X_+$ by*

$$T\left(x\right)\left(t\right) = \int_K F\left(t, s, x\left(s\right)\right) ds,\ \forall \mathrm{t} \in \mathrm{K}.$$

Then the operator T is completely continuous (see [1]). If

$$x \neq 0 \Rightarrow F\left(t, s, x\right) > 0,\ \forall \mathrm{t} \in \mathrm{K},\ \forall \mathrm{s} \in \mathrm{K},$$

and the mapping $x \mapsto F(t,s,x) > 0$ is weak-homogeneous and strictly-increasing, for every $(\mathrm{t,s}) \in K \times K$, then the operator T become weak-homogeneous and strongly-increasing. Indeed, if $x, y \in X_+$ from $x \lneqq y$ it follows that:

$$F(t,s,x(s)) \leq F(t,s,y(x)),\ \forall (\mathrm{t,s}) \in K \times K,$$

and there is the point s for which we have $x(s) < y(s)$, so for such points we have $F(t,s,x(s)) \leq F(t,s,y(s))$. The function

$$\delta(t) = \frac{\int_K F(t,s,x(s))ds}{\int_K F(t,s,y(s))ds}$$

is continuous with its maximum at a point s_0. This maximum, denoted ε_0, belongs to the interval $(0,1)$, hence for this ε_0 we have $T(x) \leqq \varepsilon_0.T(y)$. Taking $\varepsilon \in (\varepsilon_0, 1)$, we have $T(x) \lneqq \varepsilon_0.T(y)$, which means that T is strongly-increasing. If we denote by e the constant function, $e : K \to \mathbb{R},\ \mathrm{e}(\mathrm{x}) = 1,\ \forall \mathrm{x} \in \mathrm{K}$, then $I_e = X$, hence it is closed and $I_e = I_{T(x)}$, for every $x \in X_+,\ \mathrm{x} \neq 0$. The conditions of the theorem are satisfied. Taking, for example, $g : K \times K \to \mathbb{R}_+$ a continuous function which does not take the value 0, and then $F(t,s,x) = g(t,s).x^r$ with $r \in (0,1)$, it follows that there is $x^* \in X_+,\ \lambda^* \in \mathbb{R}$, such that $\|x^*\| = 1,\ \lambda^* > 0$, and:

$$\int g(t,s) x^*(s)^r\, ds = \lambda^* x^*(t),\ \forall \mathrm{t} \in \mathrm{K}.$$

Corollary 1. *Let $U : E \to E$ be an operator such that some power of it, say U^k, satisfies the hypothesis of the theorem, weak-homogeneity being replaced with positive-homogeneity. Then the conclusion of the theorem holds also for the operator U.*

Proof. There is unique $x^* \in I_{x^0}$, and $\lambda^* > 0$, such that $\frac{U^{kn}(x)}{\|U^{kn}(x)\|}$ converges to x^*, for every $x \in I_{x^0}$, and $U^k(x^*) = \lambda^*.x^*$. Also, we have:

$$U^k(U(x)) = U(U^k(x)) = U(\lambda^*.x^*) = \lambda^* U(x^*).$$

The unicity implies $U(x^*) = \|U(x)\| x^*$.

Because U is continuous, it follows that:

$$\frac{U^{kn+1}(x)}{\left\|U^{kn+1}(x)\right\|}=\frac{U\left(U^{kn}(x)\right)}{\left\|U\left(U^{kn}(x)\right)\right\|}\to\frac{U\left(x^*\right)}{\left\|U\left(x^*\right)\right\|}=x^* .$$

Applying again U we obtain the ergodicity of U.

Corollary 2. *Let T be an operator as in the Theorem 1, and let $H:E_+\to E_+$ be a homeomorphism. We suppose that H and H^{-1} are ray-preserving. Then the conclusion of Theorem 1 holds for the operator $U\overset{def}{=}HTH^{-1}$, with $H\left(I_{x^0}\right)$ instead of I_{x^0}.*

Proof. Let $x\in I_{x^0}$ and $y=H(x)$. By Theorem 1,

$$\frac{T^k(x)}{\left\|T^k(x)\right\|}\to x^* .$$

Because H is continuous and ray-preserving, it follows:

$$\frac{H\left(T^k(x)\right)}{\left\|H\left(T^k(x)\right)\right\|}\to\frac{H\left(\lambda^*x^*\right)}{\left\|H\left(\lambda^*x^*\right)\right\|}=\frac{H\left(x^*\right)}{\left\|H\left(x^*\right)\right\|}\overset{not}{=}y^* .$$

On the other hand, we have:

$$\frac{U^k(y)}{\left\|U^k(y)\right\|}=\frac{\left(HTH^{-1}\right)^k(y)}{\left\|\left(HTH^{-1}\right)^k(y)\right\|}=\frac{\left(HTH^{-1}HTH^{-1}...HTH^{-1}\right)(y)}{\left\|\left(HTH^{-1}HTH^{-1}...HTH^{-1}\right)(y)\right\|}$$

$$=\frac{\left(HT^kH^{-1}\right)(y)}{\left\|\left(HT^kH^{-1}\right)(y)\right\|}=\frac{H\left(T^k(x)\right)}{\left\|H\left(T^k(x)\right)\right\|}\to y^*.$$

Also, because U is continuous and ray-preserving too, we have:

$$\frac{U^{k+1}(y)}{\left\|U^{k+1}(y)\right\|}=\frac{U\left(U^{k}(y)\right)}{\left\|U\left(U^{k}(y)\right)\right\|}=\frac{U\left(\frac{U^{k}(y)}{\left\|U^{k}(y)\right\|}\left\|U^{k}(y)\right\|\right)}{\left\|U\left(\frac{U^{k}(y)}{\left\|U^{k}(y)\right\|}\left\|U^{k}(y)\right\|\right)\right\|}$$

$$=\frac{U\left(\frac{U^{k}(y)}{\left\|U^{k}(y)\right\|}\right)}{\left\|U\left(\frac{U^{k}(y)}{\left\|U^{k}(y)\right\|}\right)\right\|}\to\frac{U\left(y^{*}\right)}{\left\|U\left(y^{*}\right)\right\|},$$

hence $\frac{U\left(y^{*}\right)}{\left\|U\left(y^{*}\right)\right\|}=y^{*}$. It follows that:

$$U\left(y^{*}\right)=\lambda^{*}.y^{*}, \tag{4}$$

where $\lambda^{*}=\left\|U\left(y^{*}\right)\right\|$.

The unicity. Let us suppose that $U(y)=\lambda.y$, with $\|y\|=1$. Putting $x=H^{-1}(y)$ we have $T(x)=H^{-1}UH(x)=H^{-1}\left(\lambda^{*}.H(x)\right)=\lambda.x$.
Hence $x=x^{*}$, and $\lambda=\lambda^{*}$. It follows that $y=H(x)=H\left(x^{*}\right)$. On the other hand, we have:

$$y^{*}=\frac{H\left(x^{*}\right)}{\left\|H\left(x^{*}\right)\right\|}=\frac{H(x)}{\left\|H(x)\right\|}=\frac{y}{\|y\|}=y,$$

and $U(y)=\lambda.y=\lambda^{*}.y^{*}=\lambda^{*}.y$, hence $\lambda=\lambda^{*}$.

Finally, we discuss a way of producing such operators.

Proposition 1. *Let* $T_1,T_2:E_+\to E_+$ *be two weak-homogeneous and strongly-increasing operators, having the homogeneity functions* k_1 *and* k_2.

i) If the functions k_1 *and* k_2 *are increasing, then* $T_1\circ T_2$ *is strongly-increasing and weak-homogeneous, having the homogeneity function* $k_1\circ k_2$.

ii) Let $k_1=k_2$, *and let* a,b *be two nonzero positive real numbers. We define*

$$(aT_1 + bT_2)(x) = aT_1(x) + bT_2(x)$$
$$(T_1 \vee T_2)(x) = T_1(x) \vee T_2(x)$$
$$(T_1 \wedge T_2)(x) = T_1(x) \wedge T_2(x).$$

These operators are weak-homogeneous and the first two are also, strongly-increasing.

iii) If there is $x^0 \in E_+$ such that $\mathrm{I}_{x^0} = \mathrm{I}_{T_1(x)} = \mathrm{I}_{T_2(x)}$, for every $x \in \mathrm{I}_{x^0}$, then $\mathrm{I}_{x^0} = \mathrm{I}_{U(x)}$, where U is any of the operators in i) or ii). In this case, the operator $T_1 \wedge T_2$ is strongly-increasing.

Proof. We have:

$$(T_1 \circ T_2)(\lambda x) = (k_1 \circ k_2)(\lambda)(T_1 \circ T_2)(x),$$

and because the function k_2 is increasing, we have $k_2(\lambda) > 0$, if $\lambda > 0$, and also:

$$\frac{(k_1 \circ k_2)(\lambda)}{\lambda} = \frac{(k_1 \circ k_2)(\lambda)}{k_2(\lambda)} \frac{k_2(\lambda)}{\lambda}.$$

It follows that the mapping $\lambda \mapsto \frac{(k_1 \circ k_2)(\lambda)}{\lambda}$ is nonincreasing. Also, $(k_1 \circ k_2)(0) = 0$, hence $T_1 \circ T_2$ is weak-homogeneous, and his homogeneity function is $k_1 \circ k_2$.

Let now $x, y \in E_+$ be such that $0 \leqq x \lneqq y$. Then there is $\varepsilon_2 \in \mathbb{R}$, $\varepsilon_2 \in (0,1)$ such that $\mathrm{T}_2(x)\varepsilon \lneqq \mathrm{T}_2(y)$, and there is $\varepsilon_1 \in (0,1)$ such that:

$$\mathrm{T}_1(\mathrm{T}_2(x)) \lneqq \varepsilon_1 \mathrm{T}_1(\varepsilon_2 \mathrm{T}_2(y)) = \varepsilon_1 k_1(\varepsilon_2) \mathrm{T}_1(\varepsilon_2 \mathrm{T}_2(y)).$$

Because k_1 is increasing, we have $\varepsilon_1 . k_1(\varepsilon_2) < 1$, hence $T_1 \circ T_2$ is strongly-increasing.

ii) Weak-homogeneity is obvious, and also the fact that $aT_1 + bT_2$ is strongly-increasing. Let $x, y \in E_+$ be such that $0 \leqq x \lneqq y$ and let $\varepsilon_i \in \mathbb{R}$, $\varepsilon_i \in (0,1)$ be such that $\mathrm{T}_1(x) \lneqq \varepsilon_1 \mathrm{T}_1(y)$, and $\mathrm{T}_2(x) \lneqq \varepsilon_2 \mathrm{T}_2(y)$.

Then:

$$(\mathrm{T}_1 \vee \mathrm{T}_2)(x) = \mathrm{T}_1(x) \vee \mathrm{T}_2(\mathrm{x}) \leqq (\varepsilon_1 \mathrm{T}_1(y)) \vee (\varepsilon_2 \mathrm{T}_2(y))$$
$$\leqq \varepsilon_0 (\mathrm{T}_1(y) \vee \mathrm{T}_2(y)),$$

where $\varepsilon_0 = \max\{\varepsilon_1, \varepsilon_2\}$.

We have $0 \lneqq T_1(y) \leqq T_1(y) \vee T_2(y)$, so $T_1(y) \vee T_2(y) \gneqq 0$. It follows that if we take $\varepsilon \in (\varepsilon_0, 1)$, then we have:

$$(T_1 \vee T_2)(x) \leqq \varepsilon_0 (T_1(y) \vee T_2(y)) \lneqq \varepsilon (T_1(y) \vee T_2(y)),$$

hence $T_1 \vee T_2$ is strongly-increasing.

iii) Because $I_{T_1(x)} = I_{T_2(x)}$ there is $\lambda \in \mathbb{R},\ \lambda > 1$, such that $T_1(x) \leqq \lambda T_2(x)$. Then we have $bT_2(x) \leqq aT_1(x) + bT_2(x) \leqq (a\lambda + b) T_2(x)$. Hence if $U = aT_1 + bT_2$ then $I_{U(x)} = I_{T_2(x)} = I_{x^0}$.

We have $T_1(x) \leqq \lambda T_1(x)$ and so $T_1(x) \leqq \lambda (T_1 \vee T_2)(x) \leqq \lambda T_1(x)$. If $U = T_1 \wedge T_2$, this means $I_{U(x)} = I_{T_1(x)} = I_{x^0}$. We take now $U = T_1 \vee T_2$. We have

$$T_1(x) \leqq (T_1 \vee T_2)(x) \leqq (\lambda T_2 \vee T_2)(x) \leqq \lambda T_2(x).$$

It follows that $I_{T_1(x)} = I_{x^0} \subseteq I_{U(x)} \subseteq I_{T_2(x)} = I_{x^0}$. Hence $I_{U(x)} = I_{x^0}$

Let $U = T_1 \circ T_2$. Because $I_{T_2(x)} = I_{x^0}$, we have $x^0 \leqq \lambda T_2(x)$, for some $\lambda \in \mathbb{R},\ \lambda > 0$. It follows $T_1(x^0) \lneqq k_1(\lambda)(T_1 \circ T_2)(x)$.

Taking $y = T_2(x)$, we have $I_{x^0} = I_{T_1(y)}$ and hence $T_1(y) \leqq \mu x^0$, for some $\mu \in \mathbb{R},\ \mu > 0$. Hence from $T_1(x^0) \lneqq k_1(\lambda)(T_1 \circ T_2)(x) \lneqq k_1(\lambda) \mu x^0$, it follows that $I_{T_1(x^0)} \subseteq I_{U(x)} \subseteq I_{x^0}$. But $I_{T_1(x^0)} = I_{x^0}$, hence $I_{U(x)} = I_{x^0}$.

It remains to show that $T_1 \wedge T_2$ is strongly-increasing. The proof is the same as for $T_1 \vee T_2$ now the condition $(T_1 \wedge T_2)(y) \neq 0$ being implied by the equality $I_{x^0} = I_{(T_1 \wedge T_2)(y)}$.

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