

RELATING THE RIEMANN HYPOTHESIS AND THE PRIMES BETWEEN TWO CUBES

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Abstract

In this paper we make an evaluation for the number of primes between two consecutive cubes, if we assume the Riemann hypothesis. There exists at least a prime between two consecutive cubes. More precisely, if we denote by $N(n)$ the number of primes between $(n-1)^3$ and n^3 , then $N(n) \approx n^2 / \log n$.

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1. Introduction

In 1852 J. Bertrand postulated that for every integer $n \geq 2$ there is a prime between n and $2n$. Bertrand could not prove his postulate, but he verified it for all $n < 3000000$. Chebyshev was the first to prove the conjecture in 1852. This result was improved afterwards. For example, in 1989, N. Costa Pereira proved that if $x \geq 485492$, the interval $\left[x, \frac{258}{257}x \right)$ contains a prime.

The problem we study here comes from the intent to prove that between two consecutive squares there is at least a prime. This conjecture has not been neither proved nor infirmed until now. L. Skula made the hypothesis that for $n > 1$, each of the sequences $n^2 + 1, n^2 + 2, \dots, n^2 + n$ and $n^2 + n + 1, n^2 + n + 2, \dots, n^2 + 2n$ contains at least a prime.

A. Schinzel made an even stronger assumption, stating that for every real number $x \geq 8$ between x and $x + (\log x)^2$ there is at least a prime.

Sierpinski observed that from a result of A. E. Ingham it results that for m fixed and n large enough (depending of m), between n^3 and $(n+1)^3$ there exists at least m primes, that represents obviously a weaker result than the one presented here.

In this paper we evaluate the number of primes between $(n-1)^3$ and n^3 .

We give now a list of the notations used:

- $\pi(x)$ is the number of primes less than or equal to x ;
- $f(x) = o(g(x))$, if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$ (where $g(x) \neq 0$ for large values of x);
- $f(x) \sim g(x)$, if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$ (where $g(x) \neq 0$ for large values of x);
- $\Gamma(s) = \int_{0+0}^{\infty} x^{s-1} e^{-x} dx$, where $s > 0$.

2. The relation between the Riemann ζ function and the distribution of primes

The Riemann ζ function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where s is the complex variable $s = \sigma + it$, $\sigma, t \in \mathbb{R}$, $\sigma > 1$.

The ζ function is analytic and has an analytic continuation for $\sigma > 0$, excepting the point $s = 1$, given by the formula

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx.$$

In $s = 1$ the ζ function has a simple pole with the residue 1.

When $\sigma < 0$ the analytic continuation of ζ is given by the formula

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s).$$

It results that all non-positive even numbers are roots for ζ , roots that are called trivial zeros.

It was proved that the non-trivial zeros of ζ are in the so called „critic strip” $0 < \text{Re}(s) < 1$.

Riemann made the hypothesis that all the zeros of ζ are on the line $\text{Re}(s) = \frac{1}{2}$, fact that is known as the „Riemann Hypothesis” and that has not been proved until now.

On the set of the zeros, an order has been introduced, as follows: $s_1 < s_2$ if $\text{Re}(s_1) < \text{Re}(s_2)$, and if the real parts are equal, then $s_1 < s_2$ if $\text{Im}(s_1) < \text{Im}(s_2)$.

Van der Lune has proved that the first 1.500.000.000 zeros of ζ have the real part $\frac{1}{2}$.

The prime number theorem states that $\pi(x) \sim \frac{x}{\log x}$. This results taking into account the fact that $\zeta(s) \neq 0$, if $\text{Re}(s) \geq 1$ and $s \neq 1$.

As new zeros of ζ are computed, we obtain more precise information for the prime distribution function π by getting finer inequalities.

For instance, after the first 25,000 zeros of ζ have been computed, in 1938, J. B. Rosser has proved that for $x \geq 55$ we have

$$\frac{x}{\log x + 2} < \pi(x) < \frac{x}{\log x - 4}.$$

In 1962, after computing the first 3,502,500 zeros Rosser and Schoenfeld have proved that

$$\begin{aligned} \pi(x) &< \frac{x}{\log x - \frac{3}{2}}, \text{ for } x > e^{\frac{3}{2}} \text{ and} \\ \pi(x) &> \frac{x}{\log x - \frac{1}{2}}, \text{ for } x \geq 67. \end{aligned}$$

3. The main result

Assuming the Riemann hypothesis, L. Schoenfeld proves that

$$|\pi(x) - \text{li}(x)| < \frac{1}{8\pi} \sqrt{x} \log x \text{ for } x \geq 2657 \quad (1)$$

where $\text{li}(x) = \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \left(\int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^{\infty} \frac{dt}{\log t} \right).$

We denote $N(n)$ the number of primes between $(n-1)^3$ and n^3 . We prove the following:

Theorem. If n is a positive integer, then $N(n) \sim \frac{n^2}{\log n}$, assuming the Riemann

Hypothesis.

Proof:

From (1) we have

$$\text{li}(x) - \frac{1}{8\pi} \sqrt{x} \log x < \pi(x) < \text{li}(x) + \frac{1}{8\pi} \sqrt{x} \log x, \text{ for } x \geq 2657.$$

It results that

$$\begin{aligned} \pi(n^3) - \pi((n-1)^3) &> \text{li}(n^3) - \frac{3}{8\pi} n\sqrt{n} \log n - \text{li}((n-1)^3) - \frac{3}{8\pi} (n-1)\sqrt{n-1} \log(n-1) = \\ &= \int_{(n-1)^3}^{n^3} \frac{dt}{\log t} - \frac{3}{8\pi} n\sqrt{n} \log n - \frac{3}{8\pi} (n-1)\sqrt{n-1} \log(n-1) > \end{aligned}$$

$$\begin{aligned}
&> \int_{(n-1)^3}^{n^3} \frac{dt}{\log t} - \frac{3}{4\pi} n\sqrt{n} \log n > \frac{n^3 - (n-1)^3}{3 \log n} - \frac{3}{4\pi} n\sqrt{n} \log n = \\
&= \frac{3n^2 - 3n + 1}{3 \log n} - \frac{3}{4\pi} n\sqrt{n} \log n = \frac{n^2}{\log n} - \frac{3n-1}{3 \log n} - \frac{3}{4\pi} n\sqrt{n} \log n,
\end{aligned}$$

for $(n-1)^3 \geq 2657$, therefore $n \geq 15$.

Thus

$$N(n) > \frac{n^2}{\log n} - \frac{3n-1}{3 \log n} - \frac{3}{4\pi} n\sqrt{n} \log n. \quad (2)$$

We evaluate the last two terms in (2).

We have

$$\begin{aligned}
\frac{3n-1}{3 \log n} &= o\left(\frac{n^2}{\log n}\right) \text{ and} \\
n\sqrt{n} \log n &= o\left(\frac{n^2}{\log n}\right).
\end{aligned}$$

Therefore, it results that

$$N(n) > \frac{n^2}{\log n} + o\left(\frac{n^2}{\log n}\right). \quad (3)$$

From (1) it also results that

$$\begin{aligned}
\pi(n^3) - \pi((n-1)^3) &< \text{li}(n^3) + \frac{3}{8\pi} n\sqrt{n} \log n - \text{li}((n-1)^3) + \frac{3}{8\pi} (n-1)\sqrt{n-1} \log(n-1) = \\
&= \int_{(n-1)^3}^{n^3} \frac{dt}{\log t} + \frac{3}{8\pi} n\sqrt{n} \log n + \frac{3}{8\pi} (n-1)\sqrt{n-1} \log(n-1) < \\
&< \int_{(n-1)^3}^{n^3} \frac{dt}{\log t} + \frac{3}{4\pi} n\sqrt{n} \log n < \frac{n^3 - (n-1)^3}{3 \log(n-1)} + \frac{3}{4\pi} n\sqrt{n} \log n.
\end{aligned}$$

So we obtained that

$$N(n) < \frac{n^2}{\log(n-1)} - \frac{3n-1}{3 \log(n-1)} + \frac{3}{4\pi} n\sqrt{n} \log n. \quad (4)$$

Evaluating the terms in (4), we get

$$\begin{aligned}
\frac{n^2}{\log(n-1)} &\sim \frac{n^2}{\log n}, \\
\frac{3n-1}{3 \log(n-1)} &= o\left(\frac{n^2}{\log n}\right) \text{ \u015fi}
\end{aligned}$$

$$n\sqrt{n} \log n = o\left(\frac{n^2}{\log n}\right).$$

Therefore, we have

$$N(n) < \frac{n^2}{\log n} + o\left(\frac{n^2}{\log n}\right). \quad (5)$$

From (3) and (5) it results that

$$N(n) \sim \frac{n^2}{\log n}. \quad \square$$

Consequence. Assuming the Riemann Hypothesis, there exists at least a prime between two consecutive cubes.

Proof: If we assume the Riemann hypothesis, we have the relation (2)

$$N(n) > \frac{n^2}{\log n} - \frac{3n-1}{3 \log n} - \frac{3}{4\pi} n\sqrt{n} \log n.$$

In order to prove that between two cubes there is at least a prime, it is enough to prove that

$$\frac{n^2}{\log n} - \frac{3n-1}{\log n} - \frac{3}{4\pi} n\sqrt{n} \log n > 0. \quad (6)$$

It is easy to see that for $n \geq 31$, we have

$$\frac{n^2}{10 \log n} > \frac{3n+1}{3 \log n}. \quad (7)$$

Therefore, it is enough to prove that

$$\frac{9n^2}{10 \log n} > \frac{3}{4\pi} n\sqrt{n} \log n. \quad (8)$$

The previous relation is equivalent to

$$\sqrt[4]{n} > \sqrt{\frac{5}{6\pi}} \log n. \quad (9)$$

Denote $x = \sqrt[4]{n}$ and $k = 4\sqrt{\frac{5}{6\pi}} \approx 2.0601$. The relation (9) is equivalent to $x > k \log x$.

Let $f : [1, \infty) \rightarrow \mathbb{R}$, $f(x) = x - k \log x$. We have $f'(x) = 1 - \frac{k}{x} > 0$ for $x > k$, so f is increasing for $x > k$; as $f(k) > 0$ it results that $f(x) > 0$ for $x > k$ and, therefore, (8) is true for $n > k^4 \approx 18.011$.

Direct verifications for $1 \leq n \leq 31$ lead us to state that between two consecutive cubes there exists at least a prime. \square

4. Remarks

Even assuming the Riemann hypothesis, from Schoenfeld's inequality (1) it does not result that between $(n-1)^2$ and n^2 there exists at least a prime because we obtain that

$$\pi(n^2) - \pi((n-1)^2) > \frac{2n-1}{2 \log n} - \frac{1}{2\pi} n \log n,$$

and the lower bound is non-positive for n large enough.

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